Boundary Sketching with Asymptotically Optimal Distance and Rotation

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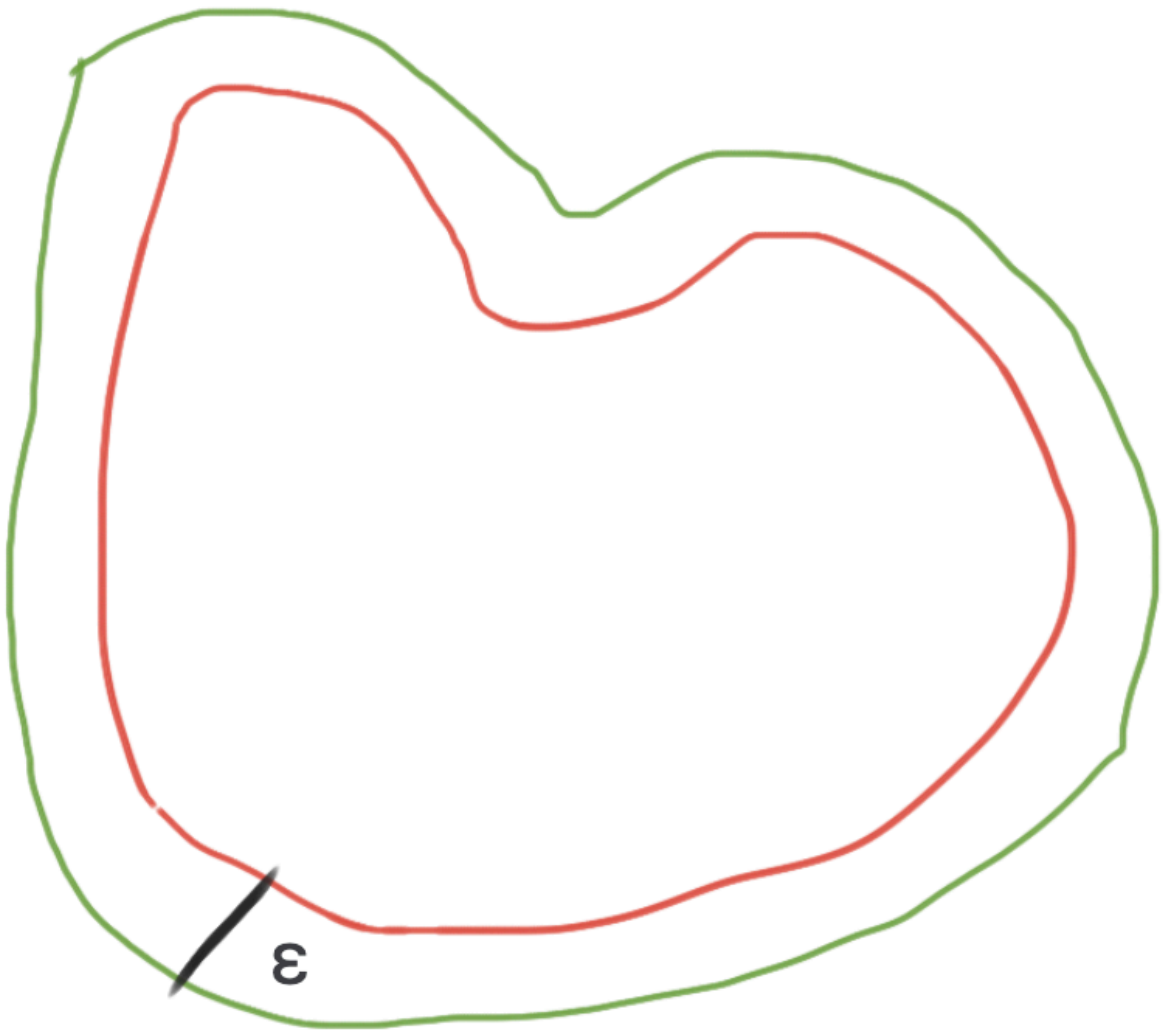
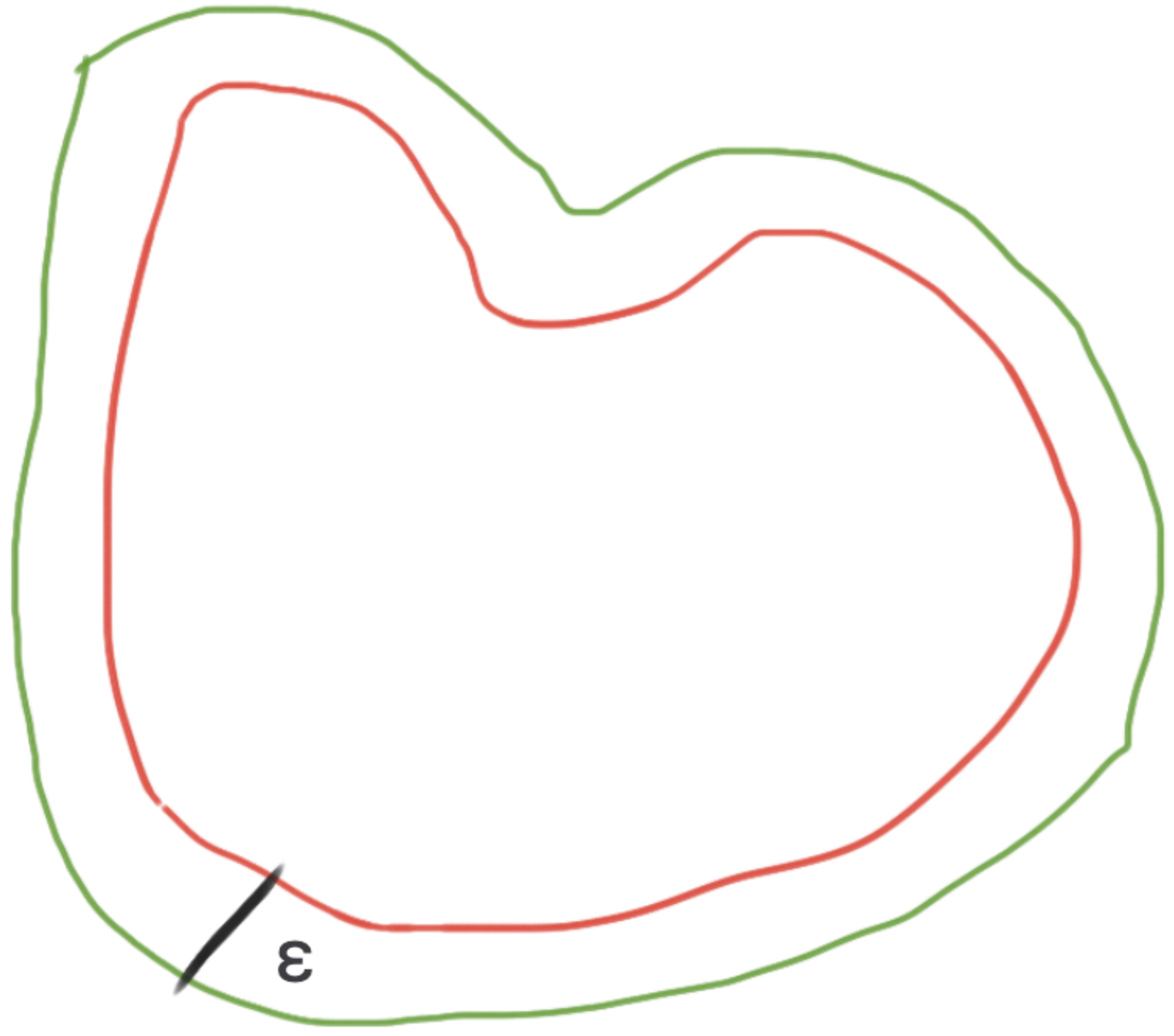
Informal Problem Model

- Robots can learn 1) if they are inside of a shape via some feedback e.g. sensor data 2) the gradient if they cross the boundary.
- **They do not have instantaneous ability to respond to feedback.**
- Can we then output a curve that is close to the shape boundary?



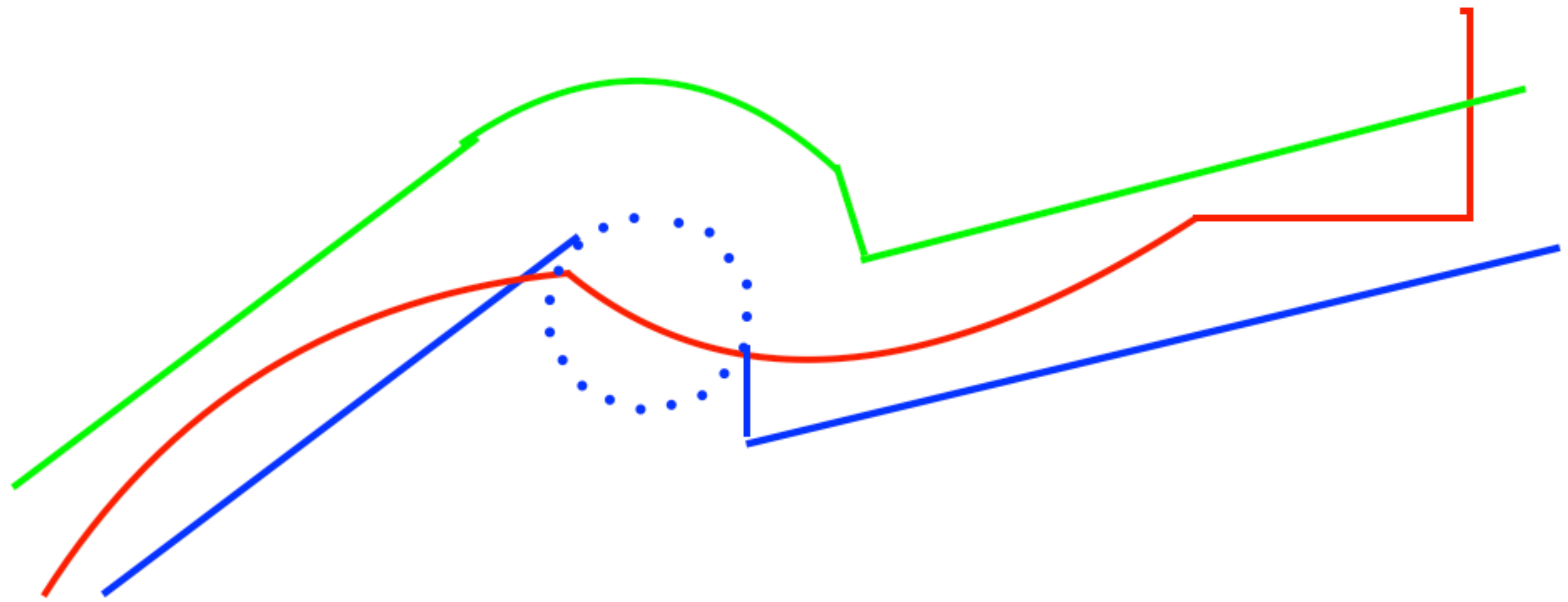
Formal Motion Constraints

- Shape boundary has unit diameter and λ is a value normalized by the diameter.
- If the robots move, they commit to a motion plan that involves either turning or traveling at least λ radians or units.
- They robots can communicate instantaneously.



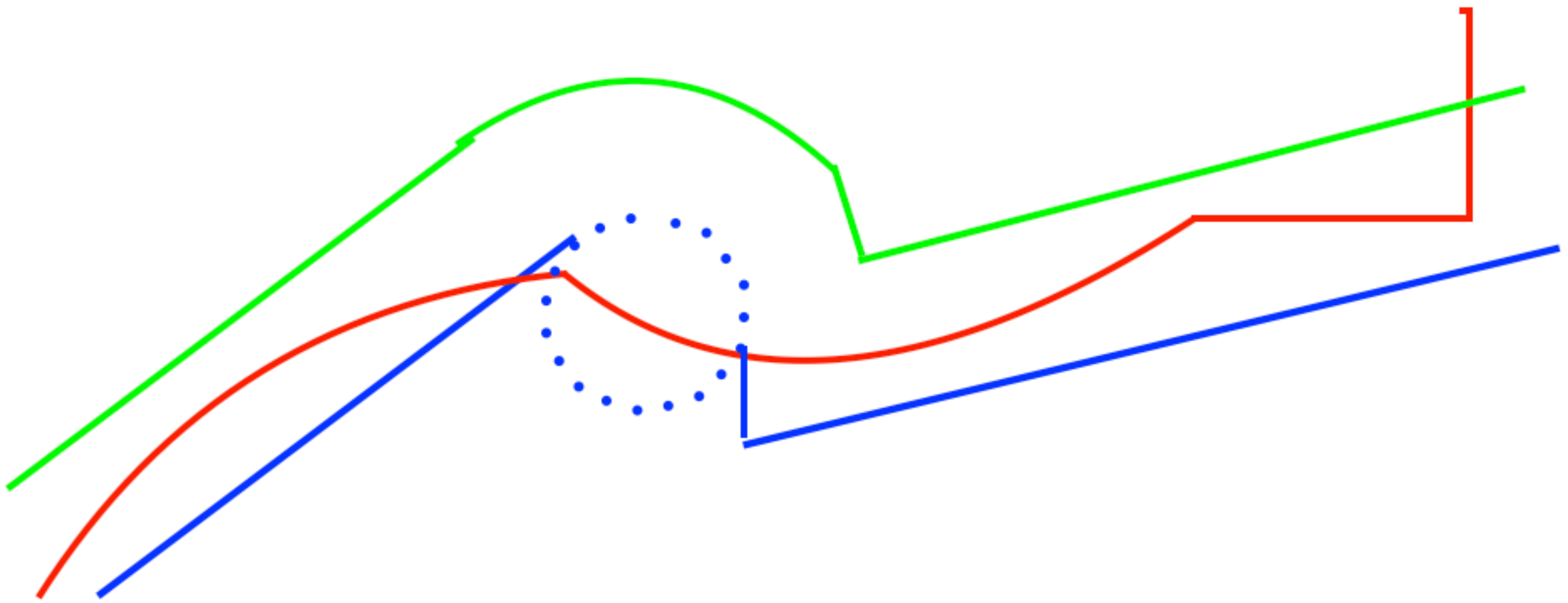
Sketch Algorithm

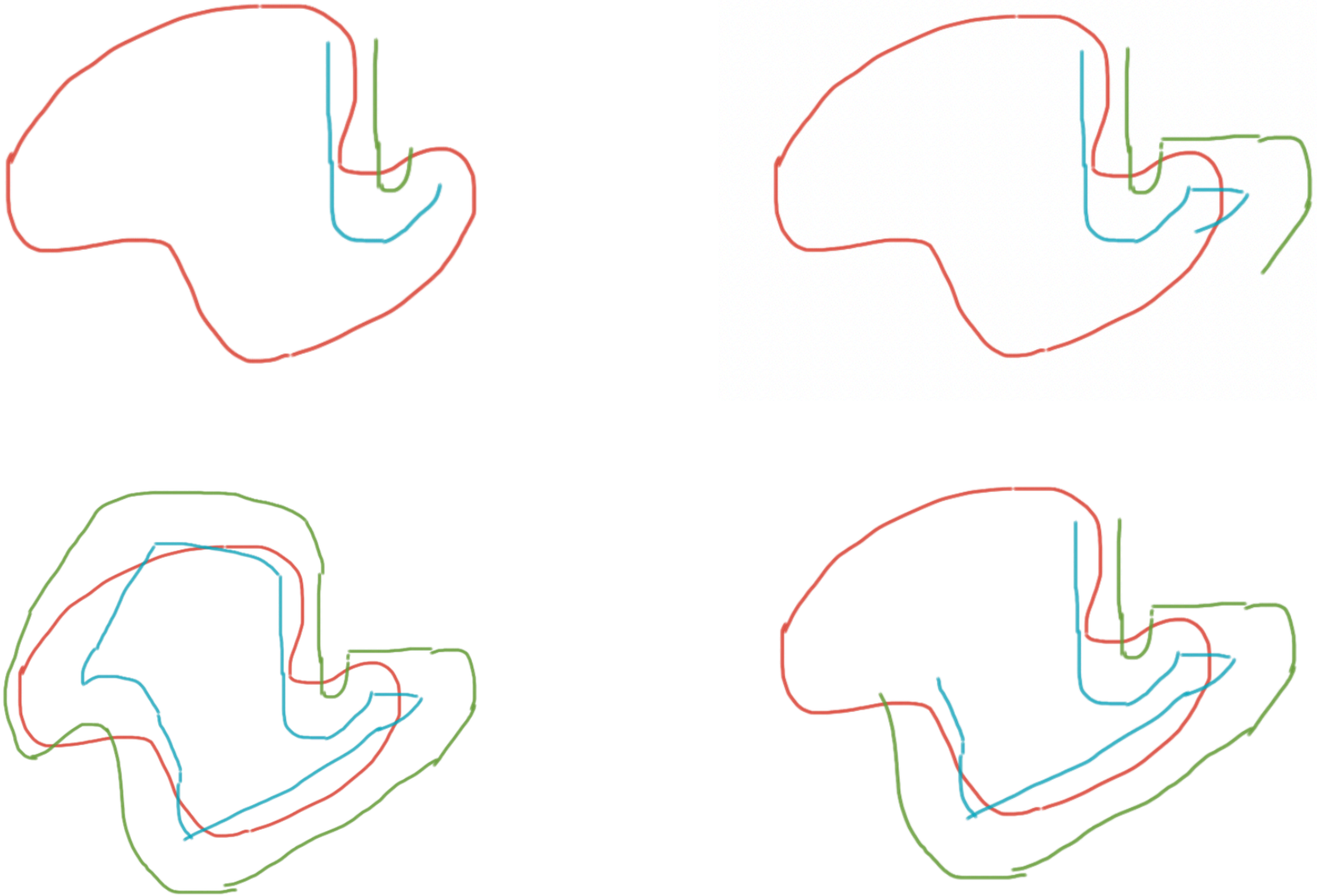
- Main Result: There exists an algorithm using two robots that can output a curve that is at most $\epsilon = 8\sqrt{\lambda}$ distance away from the input curve.
- Total radians turned and distance traversed by the robots are both asymptotically optimal.



Use of two robots

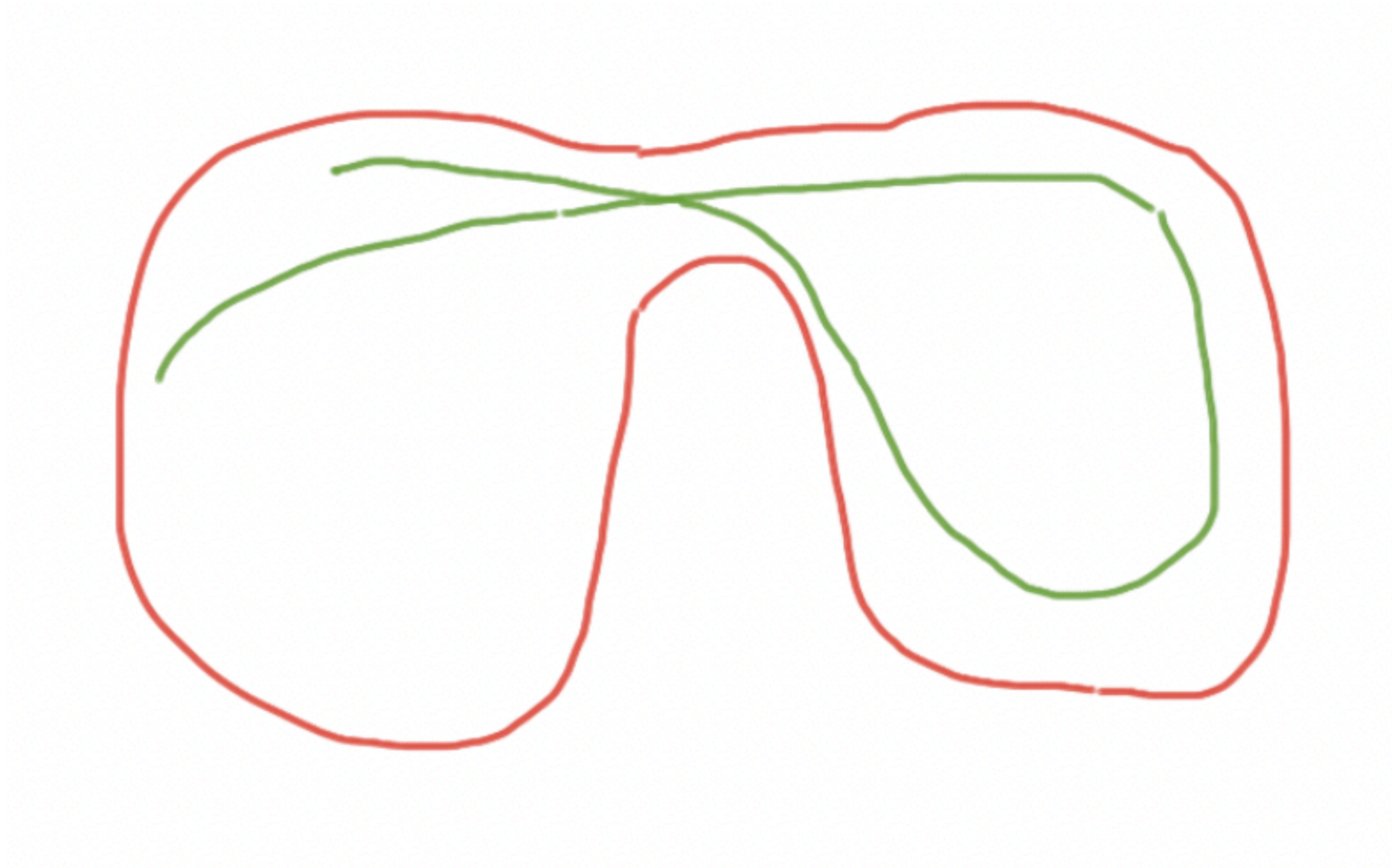
- Robot paths “sandwich” the red curve.
- When either of them cross the boundary, re-establish the “sandwich invariant”.

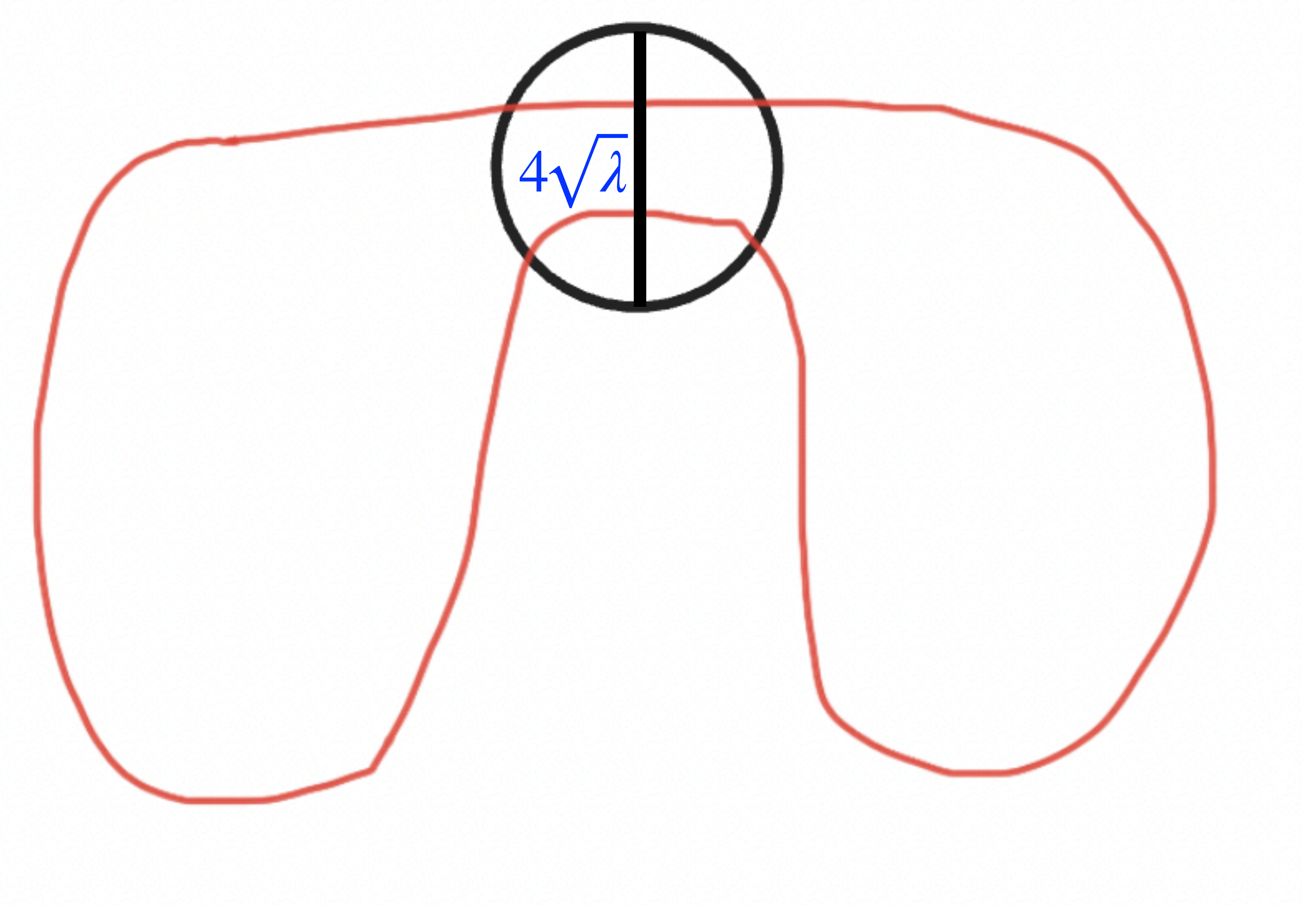




Self-Intersection

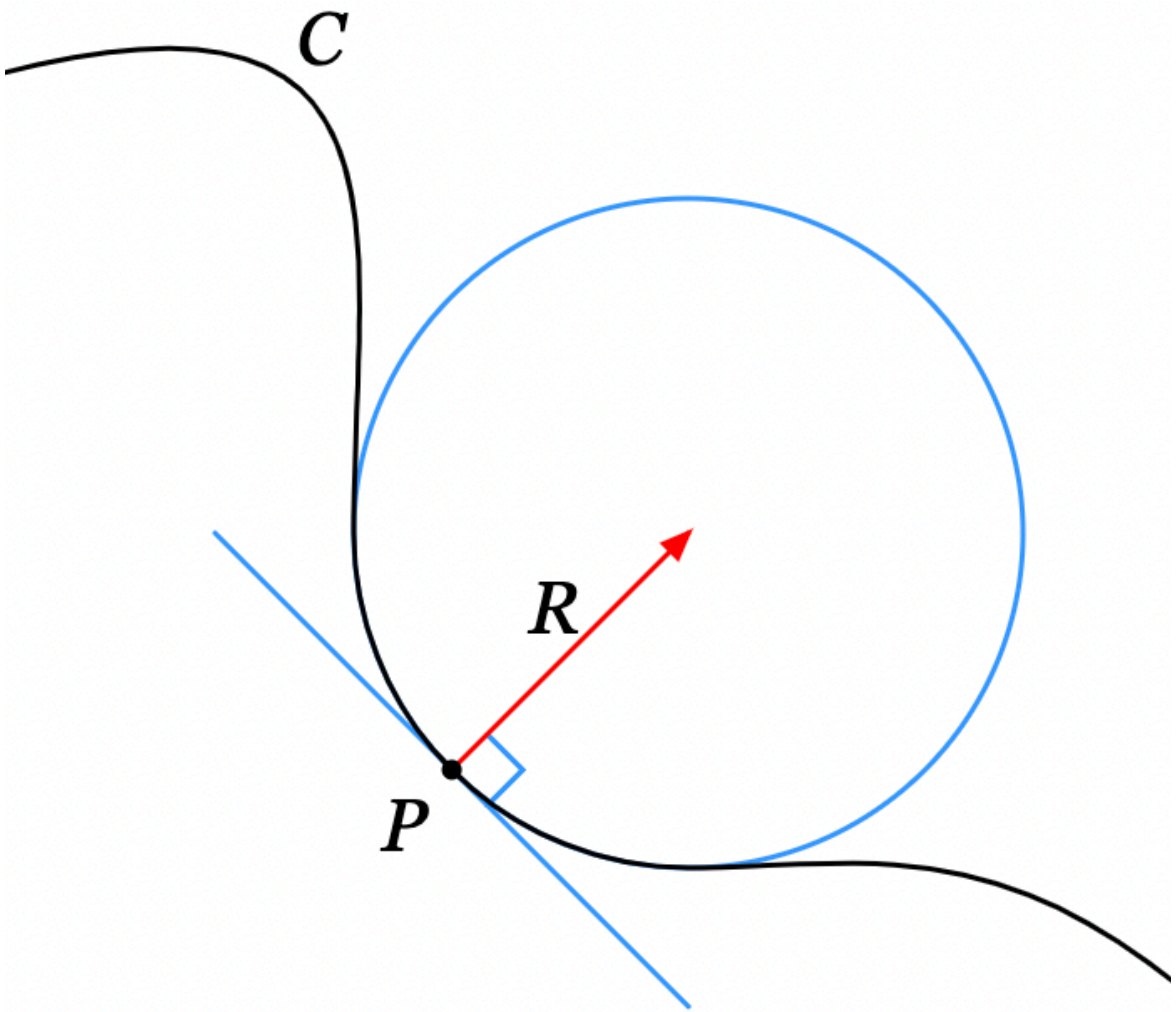
- Concern: the sketch can potentially self-intersect.
- Assume: every ball of radius $4\sqrt{\lambda}$ centered on a point on the boundary, when it intersects with the boundary has exactly one path component.
- Then the ϵ-sketch produced by the algorithm will not self-intersect.





Terminologies

- Formal notions - curvature, length, rotation from differential geometry and path component from topology.
- Key terms are ℓ, ϕ - which are length and total rotation of the input curve respectively.

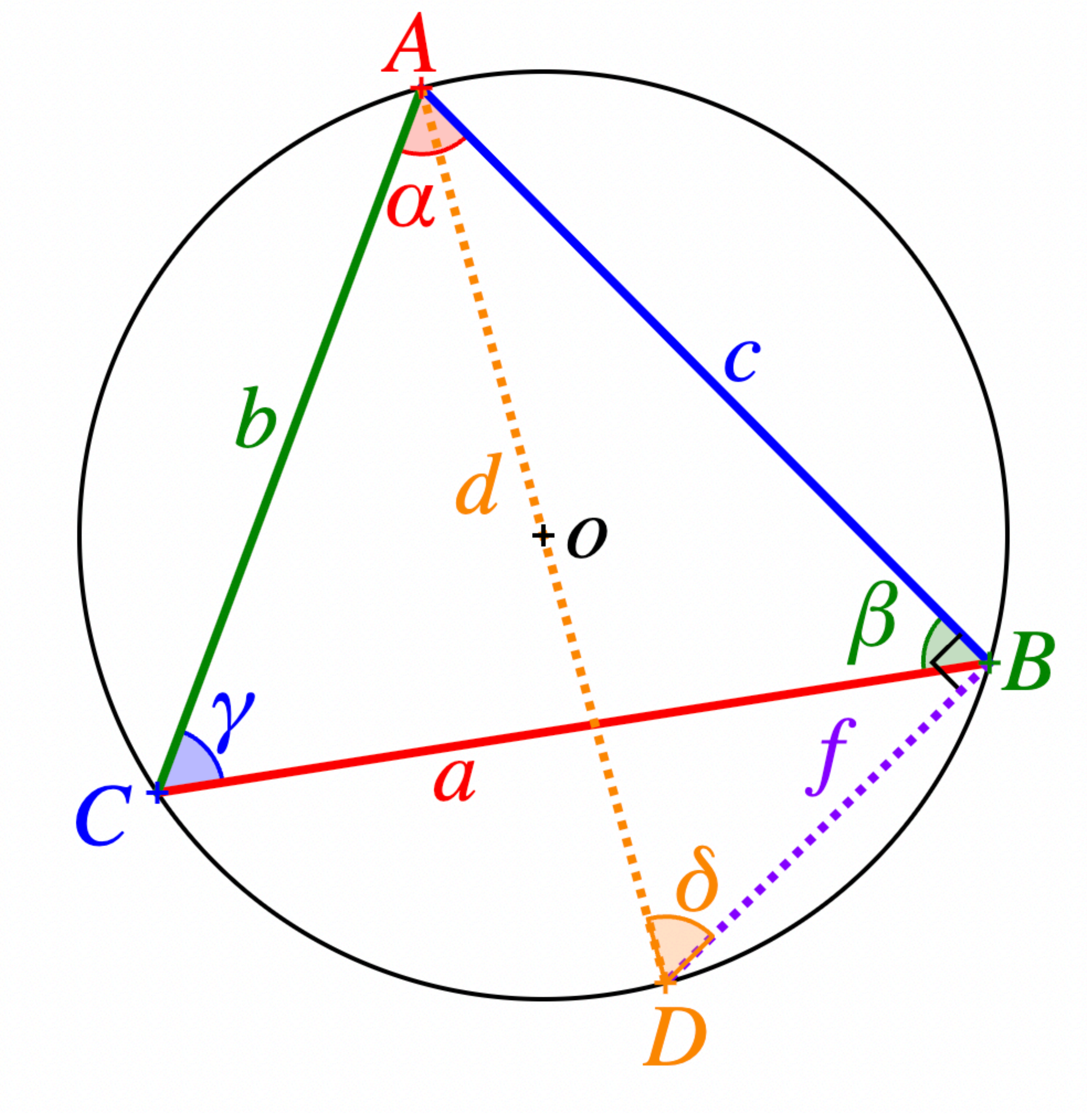


Technical Overview

- Challenges: (1) proving the output is a $\epsilon = 8\sqrt{\lambda}$-sketch (2) the sketch does not self-intersect (3) proving asymptotic optimality.
- Showing non self-intersection uses proof by contradiction to construct multiple path components.

A key lemma $\ell = O(\phi)$

- The term ℓ shows up in the analysis of total rotation.
- If the shape has unit diameter, simple to prove using law of sines for polygons.



$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

Proving $\ell = O(\phi)$

- For general curves, we construct a polygon with a number of properties.
- Key observation: polygon has finite number of endpoints where in between ζ has either a lower or upper bound on its curvature.
- To show that: use continuity on compact sets $\implies$ *uniform continuity*.

$$\phi = \int_0^{\ell(\zeta)} |\kappa(t)|\, dt$$

Proving $\ell = O(\phi)$

- Step 1: upper bound asymptotically the length of ζ between the two endpoints of polygon sides by the side length of the polygon.

- Step 2: show ζ always has some point in between the endpoints of polygon sides where its gradient points to the direction of the respective polygon side.

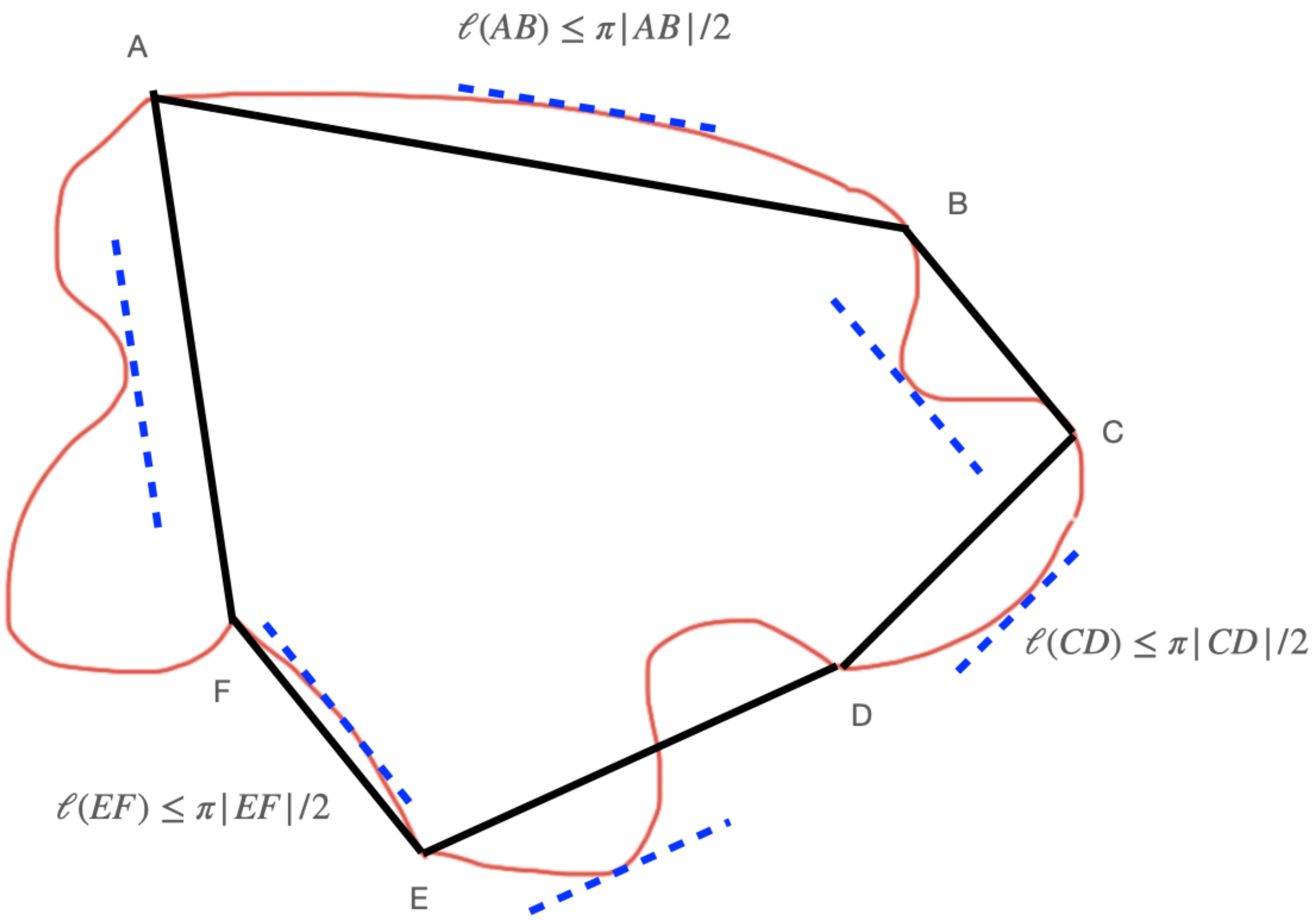
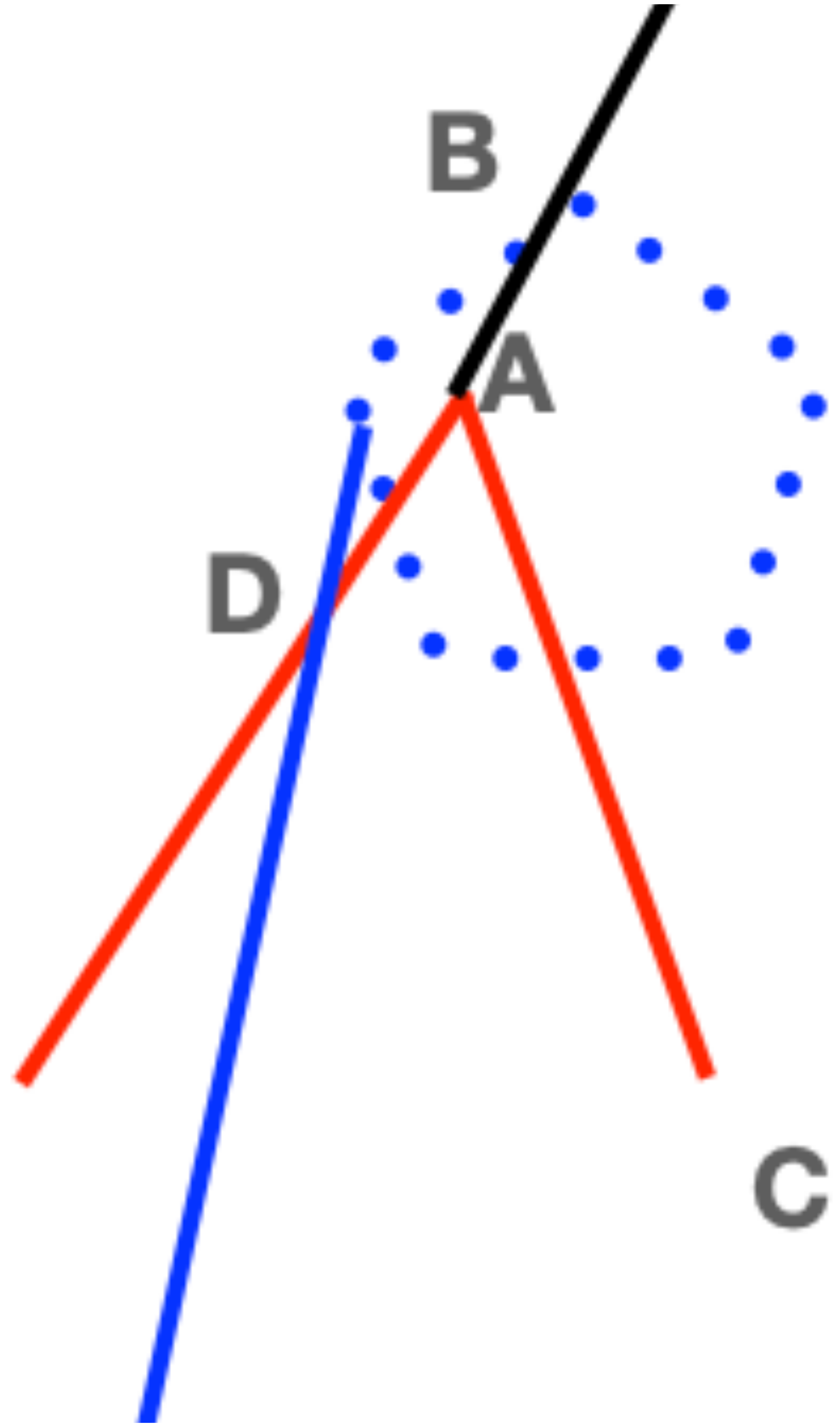


Illustration for $\ell = O(\phi)$ proof. The black line segments show the constructed polygon for the shape in red, dotted blue lines are tangents that are parallel to corresponding polygon sides, the inequalities compare the length of the shape against the length of the polygon side.

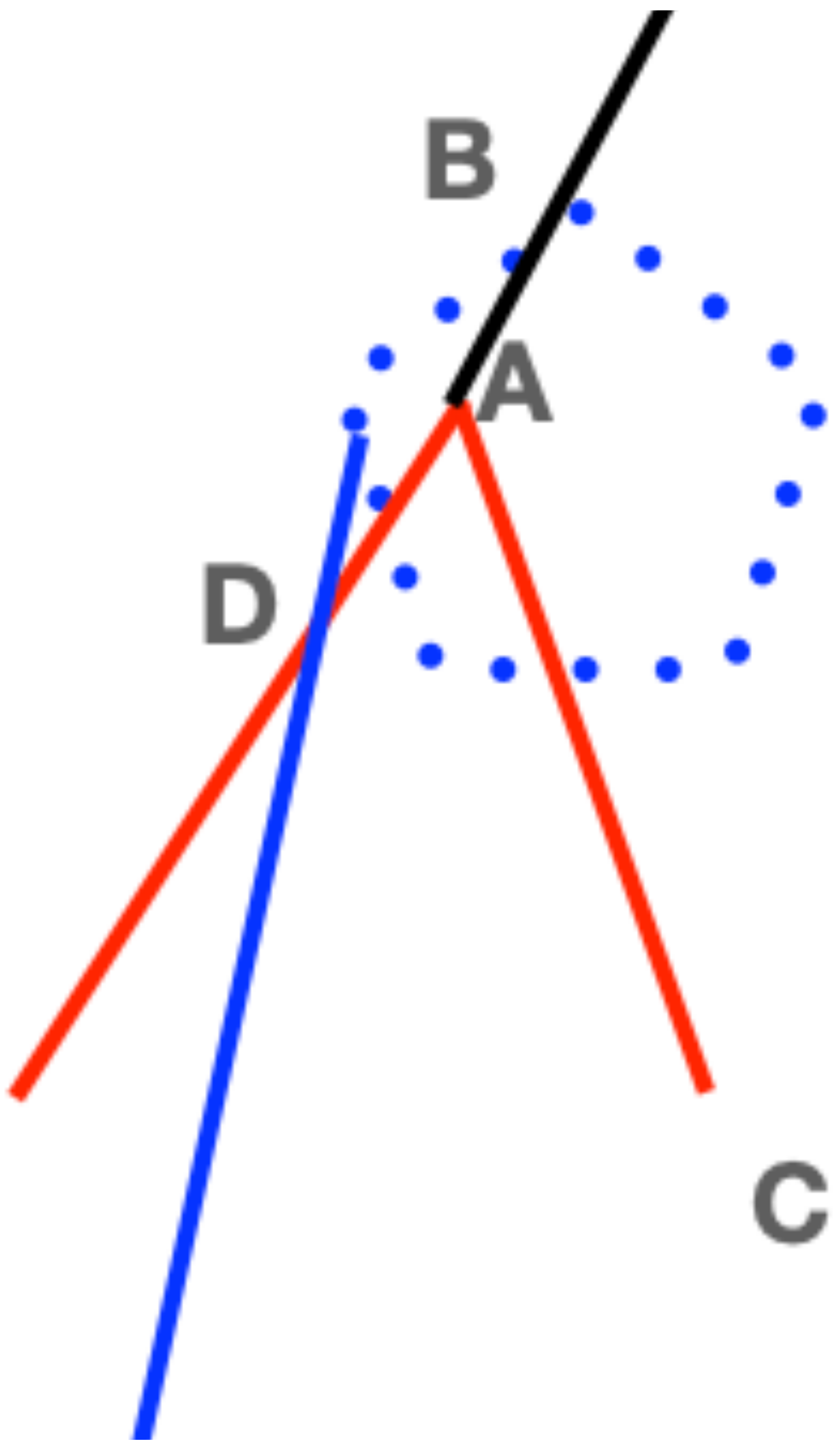
Asymptotic Analysis

- The robots rotate $\sqrt{\lambda}$ rather than λ after crossing the boundary. This guarantees the re-establishment of sandwich invariant happens fast enough.
- In particular, the number of crossings is shown to be $O(\ell/\sqrt{\lambda})$.



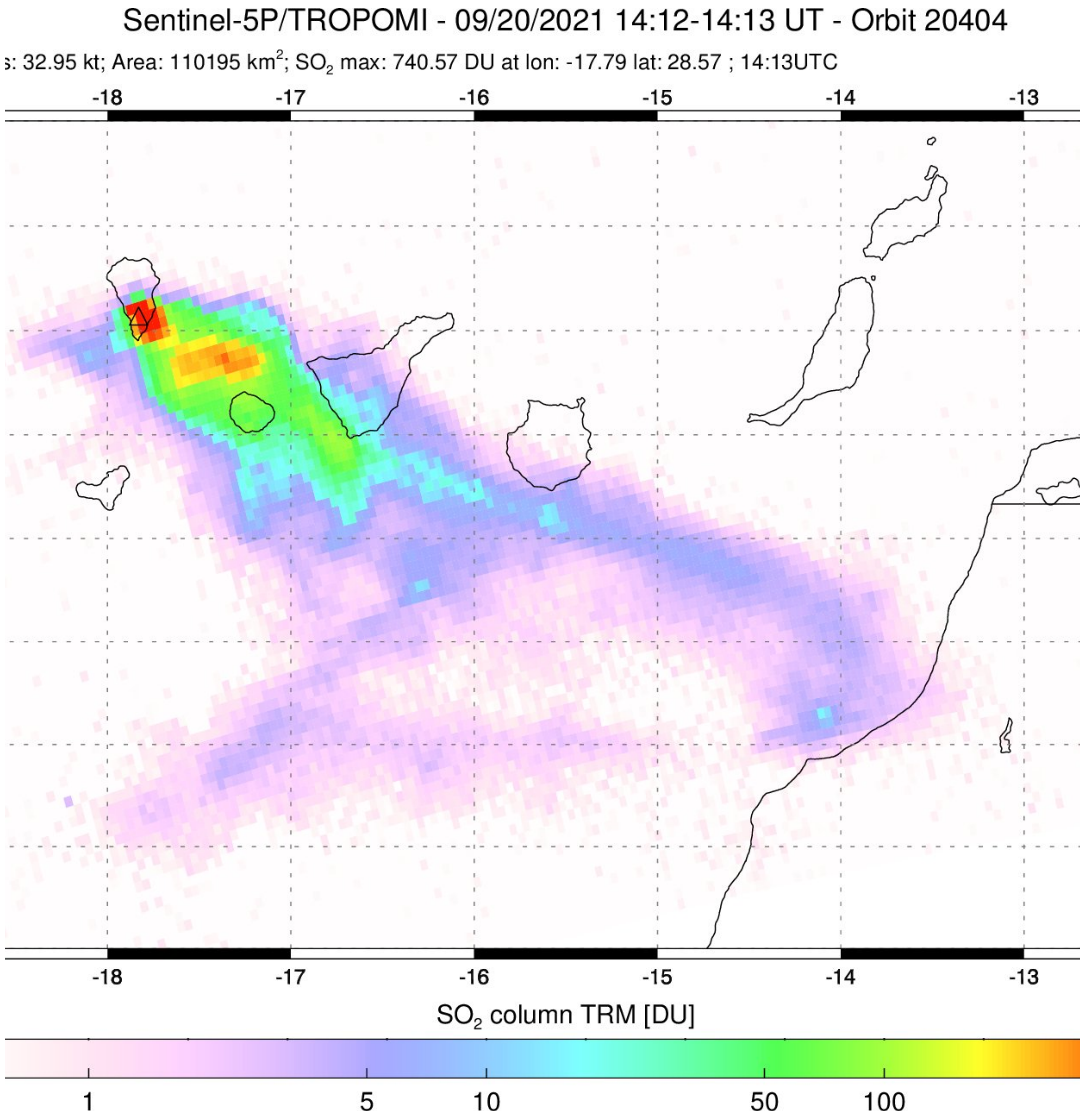
Asymptotic Analysis

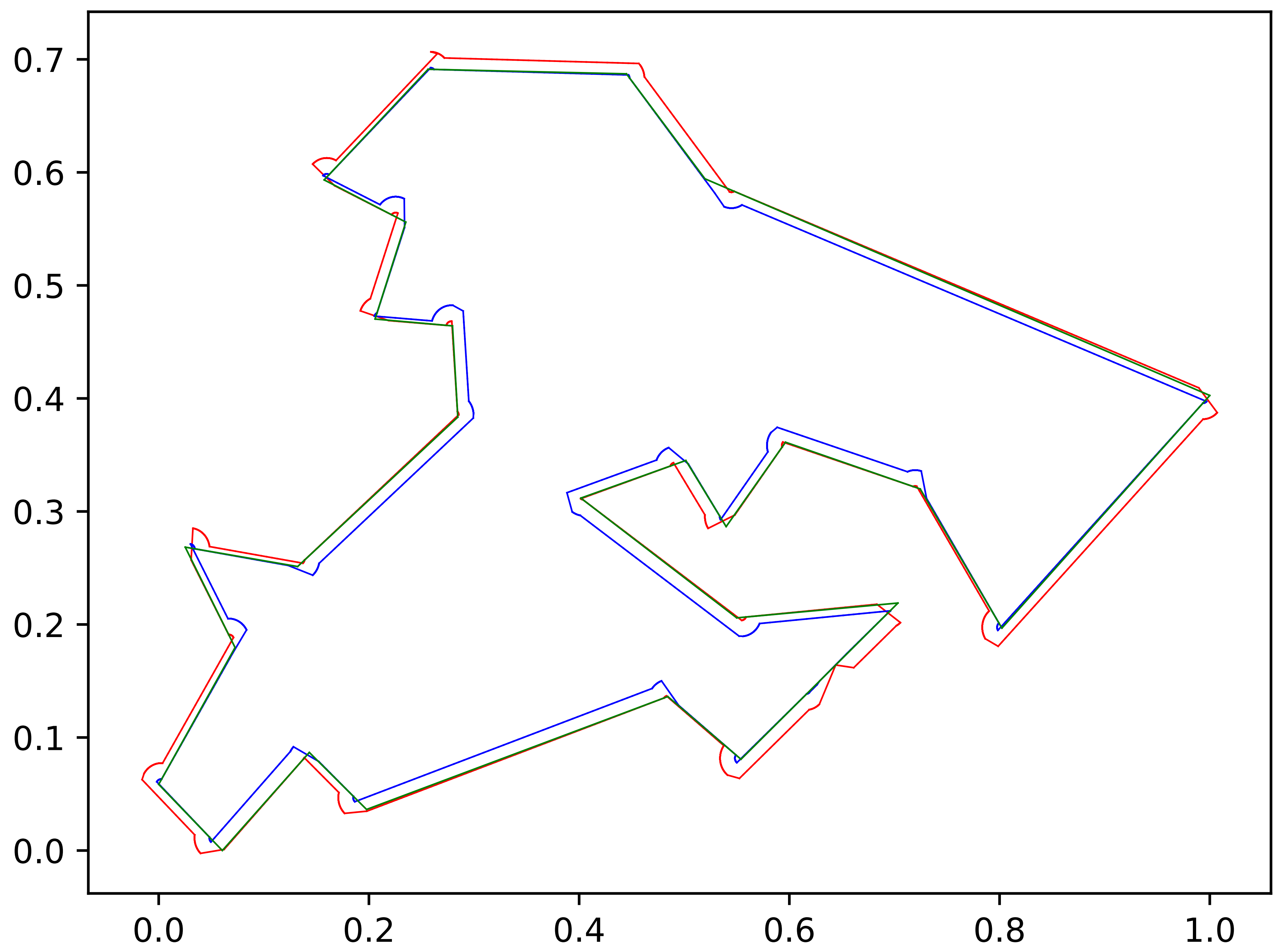
- One can bound the number of steps a robot takes after crossing in relation to the amount ζ rotates.
- In addition to this bound, one utilizes the number of crossings $O(\ell/\sqrt{\lambda})$, hence the appearance of ℓ in rotation asymptotic.



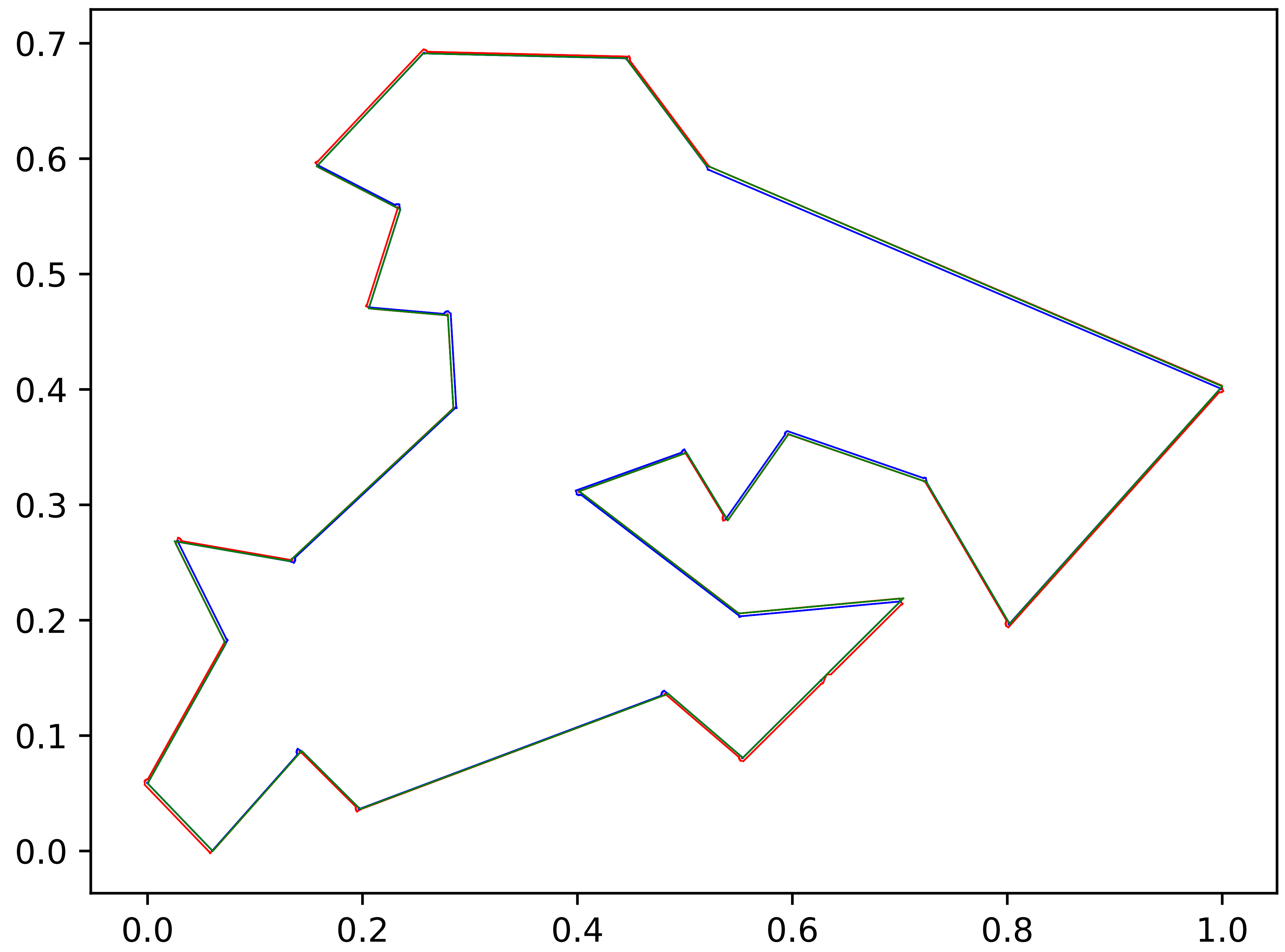
Simulations

- Plume shape approximated by a polygon or gaussian.
- Gradient for polygons therefore is the slope of the sides of the polygon.
- Gradient for gaussians estimated using least squares error.
- Simulation conducted for different values of λ.





$\lambda = 0.000025$



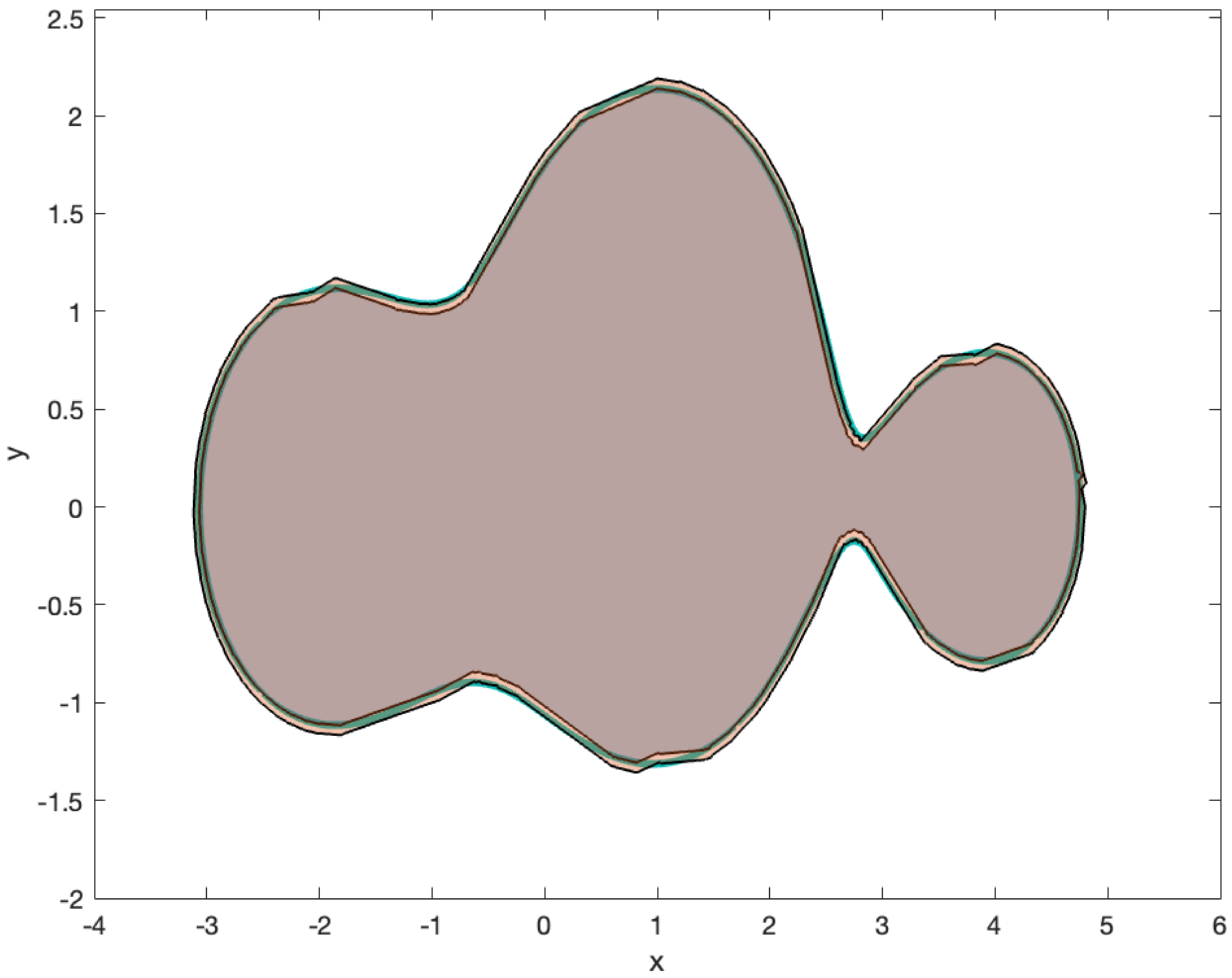
$\lambda = 0.000001$

Estimating Gradient

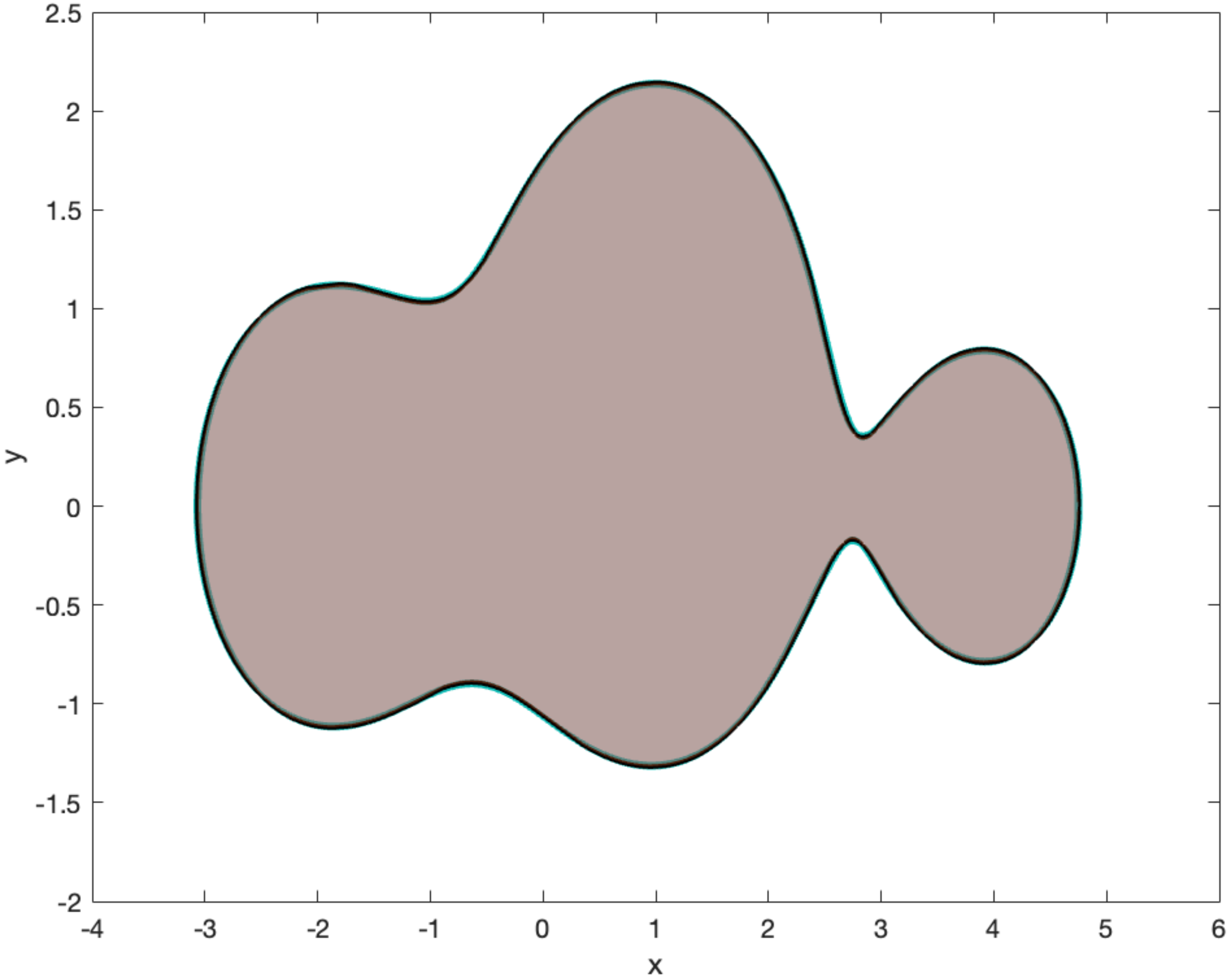
- Consider three points a, b, c that are close together, and we measure $f(a), f(b), f(c)$.
- a, b, c are successive points of a robot's path with the middle one being where it crosses the shape boundary.
- The estimated gradient $\nabla f(b) = (x', y')$ is a vector that minimizes the following sum:
- $(f(a) - f(b) - \nabla f(b) \cdot (a - b))^2 + (f(c) - f(b) - \nabla f(b) \cdot (c - b))^2$

Estimating Gradient

- Setting the rows of a 2×2 matrix A to be $a - b, c - b$ respectively and the entries of vector β to be $f(a) - f(b), f(c) - f(b)$ respectively, this problem is formally solving for the vector v such that, $|Av - \beta|^2$ is smallest.
- We can utilize a pseudo-inverse based method for solving this LSQ.



$\lambda = 0.0025$



$\lambda = 0.0001$

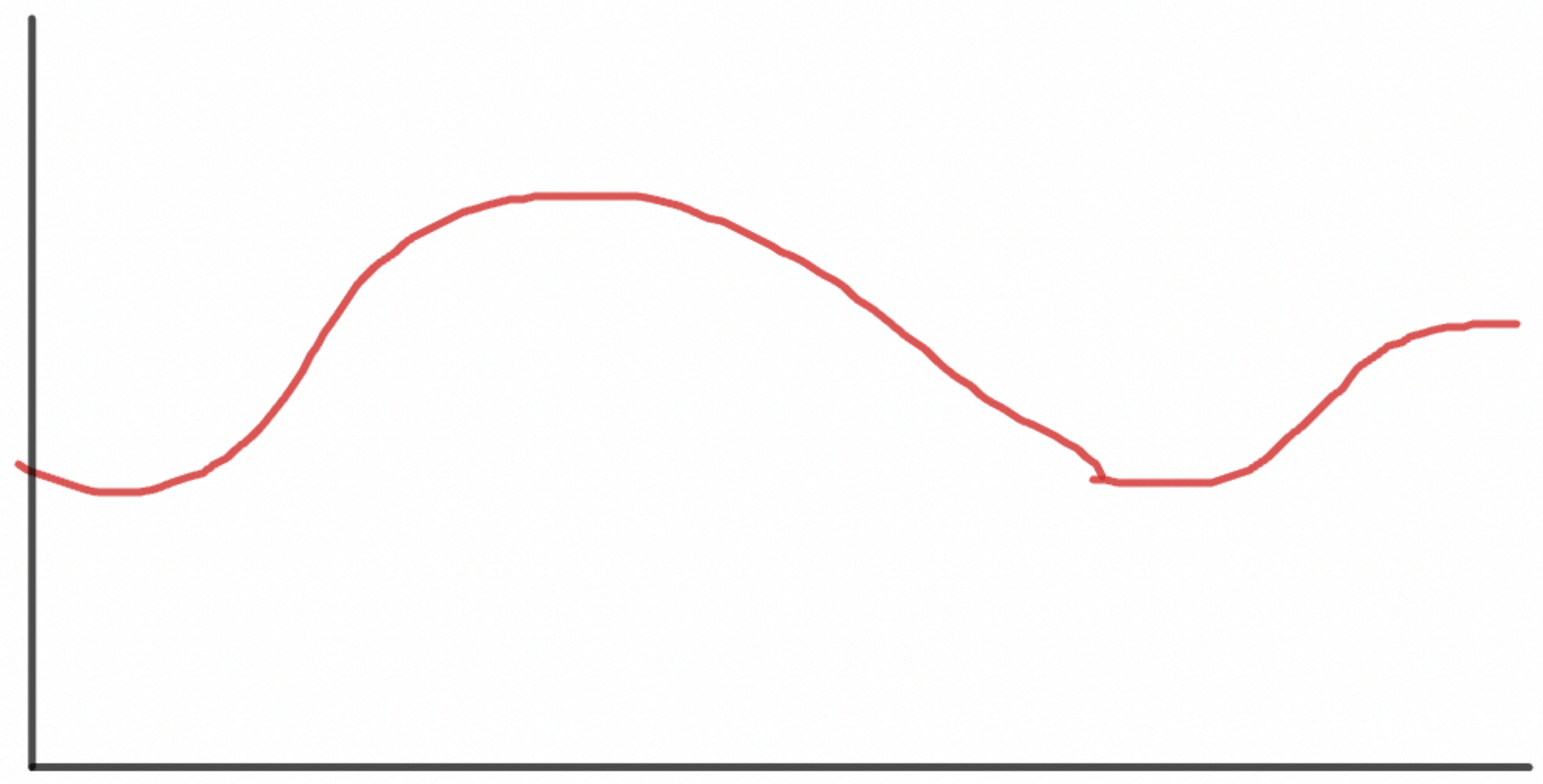
Conclusion

- We presented a distributed algorithm using two robots to track the boundary of a shape that uses *only* local information at a time.
- The robots are not assumed to be capable of responding instantaneously to external feedback.
- The asymptotic distance traversed and angle turned are both optimal.

Future Work

- Is it possible to do boundary sketching using *only* ***one*** *robot?* We conjecture that the answer is no (via boolean functions and Fourier analysis).

- Is it possible to reduce the estimation error from $\epsilon = 8\sqrt{\lambda}$ to $\epsilon = O(\lambda)$?



Questions?

Appendix : Synchronization