On the Power of Threshold-Based Algorithms for Detecting Cycles in the CONGEST model

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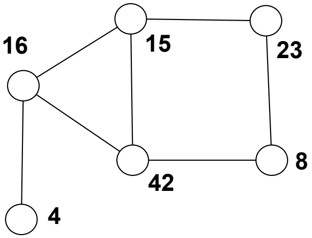
Reminder: CONGEST model

- $G = (V, E)$. Denote $n = |V|$.
- Arbitrary identifiers $\in \text{poly}(n)$.
- Synchronous rounds.
- $O(\log n)$ bits of communication per edge per round.



$n = 6$

C_p-freeness

A p-cycle is a cycle of length p.

C_p-freeness problem:

- If G contains a p-cycle, at least one node must reject.
- All nodes accept otherwise.

State of the art

Length	Round complexity	Ref.
4	$\tilde{\Theta}(\sqrt{n})$	[DKO'14]
$2k+1 \geq 5$	$\tilde{\Theta}(n)$	[KR'17]
$2k$	$O(n^{1-1/(k(k-1))})$	[EFGKO'18]
$\mathbf{2k = 6, 8, 10}$	$\mathbf{O(n^{1-1/k})}$	**[CFGLLO'20]**

Our work

Threshold-based algorithm [CFGLLO'20]: Best complexity proven for $k \leq 5$.

Does it work for bigger lengths ? **No.**

Theorem

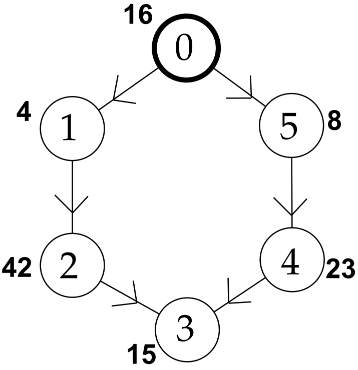
For any $k \geq 6$, threshold-based algorithms cannot solve C_{2k}-freeness in sublinear round complexity.

Reminder: Color-coding

Color-coding: Decide if a specific node u_0 is in a $2k$-cycle in $\tilde{O}(1)$ rounds.

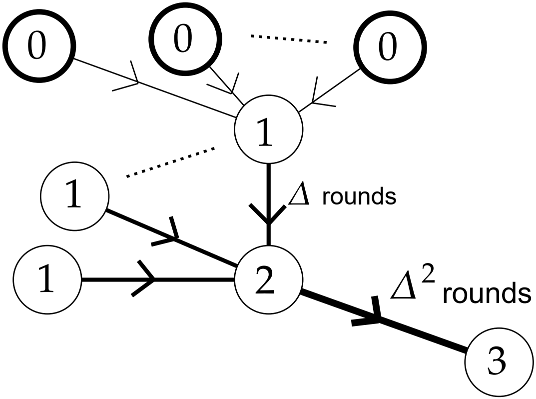
Repeat $\tilde{\Theta}(1)$ times:

1. Nodes $\leftarrow$ color u.a.r. in $[0, 5]$.
2. If u_0 colored 0: sends ID.
3. Received ID forwarded.
   from color 1 to 2 to 3.
   and from 5 to 4 to 3.
4. Color 3 detects cycle.



Limit of color-coding

Drawback: Parallelization for several starter nodes → CONGESTION.



In general, $O(\Delta^{k-1})$ rounds for $2k$-cycle.

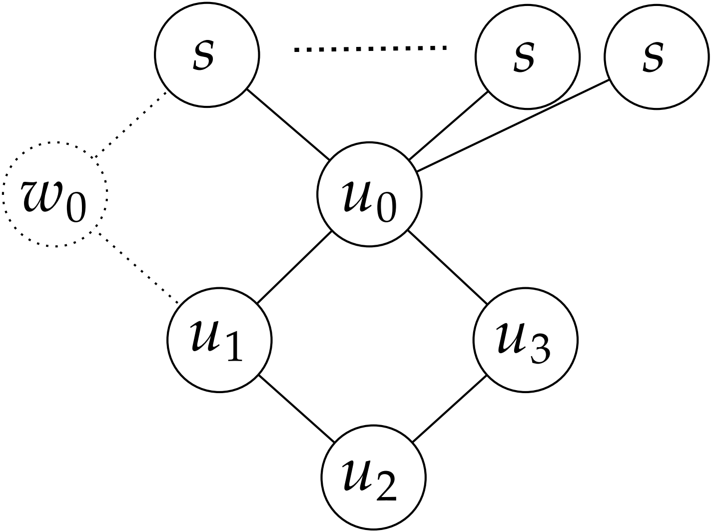
Threshold-based algorithm

WIN-WIN situation: only small degree $\rightarrow$ *limited congestion*.

- Color-coding only with nodes of degree $\leq n^{1/k}$.
- Complexity: $O(n^{\frac{k-1}{k}}) = O(n^{1-1/k})$ rounds

- **To do:** $2k$-cycle with node of degree $> n^{1/k}$ $\rightarrow$ *quick to sample neighborhood*.

Example for C_4



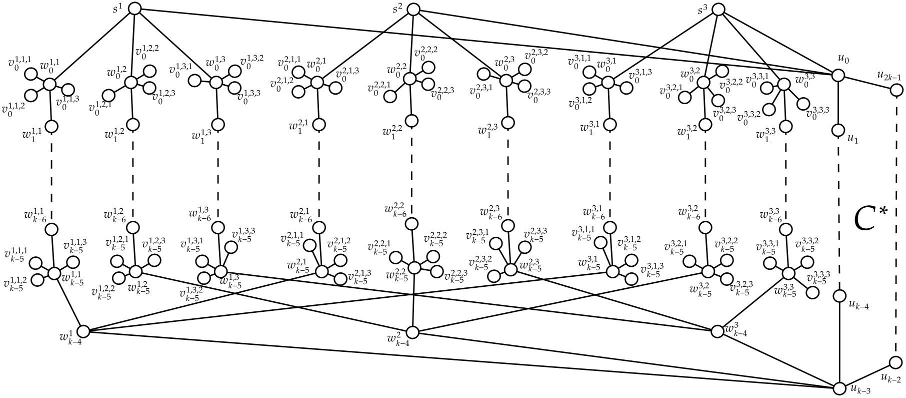
Threshold-based algorithm

Cycle with node u_0 of degree $> n^{1/k}$. Repeat $\Theta(n^{1-1/k})$ times:

1. Draw node s u.a.r. $\rightarrow$ *probability* $1/n^{1-1/k}$ *of neighboring* u_0
2. Color-coding from s, reject if in $2k$-cycle
3. Neighbors of s launch their own color-coding.
   Threshold: if node colored i receives $> T$ identifiers, it does not forward. $\rightarrow$ *NO CONGESTION !*

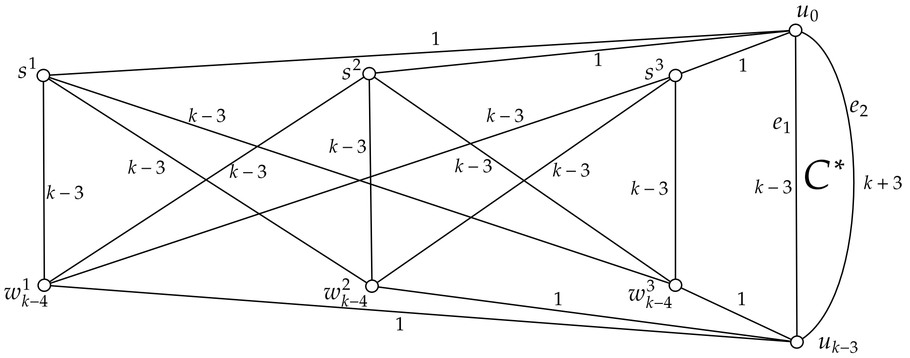
Intuition: neighbors of u_0 which are not in a $2k$-cycle of their own will *usually* respect the threshold.

For any $k \geq 6$, threshold-based algorithms cannot solve C_{2k}-freeness in sublinear round complexity.



Impossibility

Simplified graph.



Positive results

Different lengths can work together to mutually fill their *"proof holes"*.

$\{C_{12}, C_{14}\}$-freeness: neighbors of u_0 which are neither in a 12 or 14-cycle will *usually* respect the threshold.

Similar, more general result for bigger families of cycles.

Conclusion

- **Short-term:** are there other small families whose freeness threshold-based algorithms can solve ?

- **Long-term:** is there an algorithm that can solve C_{2k}-freeness for $k \geq 6$ in $O(n^{1-1/k})$?

Thank you !