The energy complexity of diameter and minimum cut computation in bounded-genus networks

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The radio network model

- A **multi-hop network** of is modeled as a graph $G = (V, E)$:
  - Each vertex $v \in V$ is a device.
  - $\{u, v\} \in E$ if u and v are within the transmission range of each other.
  - $n = |V|$ is the number of devices.

- **Synchronized communication:**
  - Time is divided into discrete slots.

The radio network model

- Each device at each time slot chooses to do one of the following actions:
  - **Idle** – do nothing.
  - **Transmit** – send a message.
  - **Listen** – listen to the channel.

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If more than one device in the neighborhood $N(v)$ of v transmit in the same time slot, then a collision occurs.

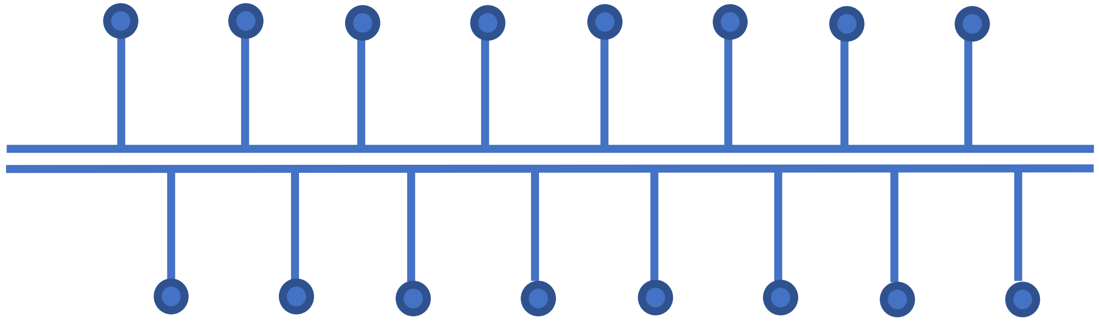
For each listening device v, it successfully receives a message from $u \in N(v)$ if u is the only transmitting device in $N(v)$.

Time & energy complexity

- Two main complexity measures:
  - **Time** – number of communication rounds (time slots).
  - **Energy** – number of channel accesses (listen and transmit).

Prior work: single-hop networks

- Most of the early work on the energy complexity focused on single-hop radio networks:
  - The special case where $G = (V, E)$ is a complete graph.
- Over the last two decades, there has been a long line of research to optimize the energy complexity of leader election and its related problems.



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Nakano and Olariu, ISAAC 2000

Jurdzinski, Kutylowski, and Zatopianski PODC 2002

Kutylowski and Rutkowski, ESA 2003

Lavault, Marckert, and Ravelomanana, Information and Computation 2007

Bender, Kopelowitz, Pettie, and Young, STOC 2016

Chang, Kopelowitz, Pettie, Wang, and Zhan, STOC 2017

Chang, Duan, and Jiang, SPAA 2021

Chang and Jiang, SPAA 2022

… and many more

Prior work: multi-hop networks

- This line of research was recently extended to multi-hop radio networks.

The energy complexity of broadcast
- Chang, Dani, Hayes, He, Li and Pettie
- PODC 2018

- **Upper bound:** A polylogarithmic-energy algorithm for broadcasting a message from a vertex to the entire network.
- **Lower bound:** Sending a message from one endpoint of a path to the other endpoint costs $\Omega(\log n)$ energy.

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Chang, Dani, Hayes, and Pettie, PODC 2020 — BFS with $n^{o(1)}$ energy

Dani and Hayes, DISC 2022 — BFS with polylog n energy

↓

An approximation $\widetilde{D}$ of the diameter D such that $\left\lfloor \frac{2D}{3} \right\rfloor \leq \widetilde{D} \leq D$ can be computed with $\tilde{O}(\sqrt{n})$ energy

Dani, Gupta, Hayes, and Pettie, DISC 2021 — Maximal matching with polylog n energy

Prior work: lower bounds

- Not all problems admit energy-efficient algorithms in multi-hop radio networks.

Chang, Dani, Hayes, and Pettie, PODC 2020

  - Computing a $(2-\epsilon)$-approximation of the diameter requires $\Omega(n)$ energy.
  - Computing a $(1.5-\epsilon)$-approximation of the diameter requires $\tilde{\Omega}(n)$ energy in sparse graphs:

The lower bound holds for graphs with $O(\log n)$ treewidth and $O(\log n)$ arboricity.

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To improve energy efficiency for diameter computation, it is necessary that we focus on a special graph class.

Our focus: bounded-genus graphs

- The **genus** of a graph G is the minimum number g such that G can be drawn on an oriented surface of g handles without crossing.
  - Planar graphs $=$ the graphs with genus zero.
  - Graphs that can be drawn on a torus without crossing $=$ the graphs with genus at most one.
- A class of graphs is **bounded-genus** if the genus of all graphs in the class can be upper bounded by some constant.

Our results

Problem	Time	Energy
Diameter	$\tilde{O}(n^{1.5})$	$\tilde{O}(\sqrt{n})$
Minimum cut	$\tilde{O}(n^{1.5})$	$\tilde{O}(\sqrt{n})$
$(1+\epsilon)$-approximate s-t minimum cut	$\tilde{O}(n^{1.5}) + \tilde{O}(\sqrt{n}) \cdot \epsilon^{-O(1)}$	$\tilde{O}\left(\sqrt{n} + \epsilon^{-O(1)}\right)$

These algorithms apply to any bounded-genus graphs.

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$\Omega(n)$ energy lower bound:

- Minimum cut of a unit-disc graph.
- s-t minimum cut of a planar graph.

Our approach

- The starting point:
  - The topology of a graph with maximum degree Δ can be learned using $\tilde{O}(\Delta)$ energy.

  Chang, Dani, Hayes, He, Li and Pettie, PODC 2018

- $V_H =$ the set of vertices whose degree is at least $\sqrt{n}$.
- $V_L = V \setminus V_H$.

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G has bounded genus → G has bounded arboricity → $|V_H| = O(\sqrt{n})$.

↓

The topology of the subgraph induced by V_H can be learned with $\tilde{O}(\sqrt{n})$ energy.

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- Let S be a connected component of $G[V_L]$.

⬇

All vertices in S can learn the topology of the subgraph induced by S with $\tilde{O}(\sqrt{n})$ energy.

Our approach

Observation: If the number of connected components of $G[V_L]$ is $\tilde{O}(\sqrt{n})$, then we are done:

- The entire graph topology of G can be learned with $\tilde{O}(\sqrt{n})$ energy via $\tilde{O}(\sqrt{n})$ invocations of a broadcast algorithm.

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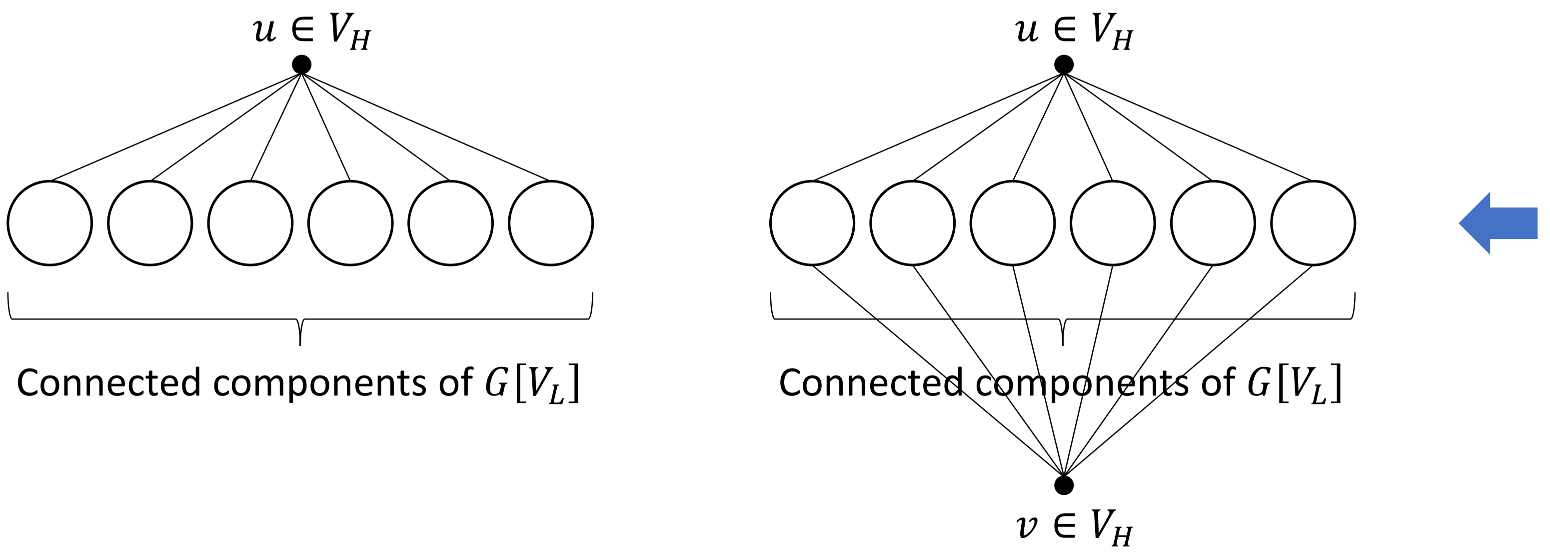
- The number of connected components of $G[V_L]$ can be $\Omega(n)$.



These structures can be realized as planar graphs.

Our approach

Observation: If the number of connected components of $G[V_L]$ is $\tilde{O}(\sqrt{n})$, then we are done.

- $V_H =$ the set of vertices whose degree is at least $\sqrt{n}$.
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Claim: The number of **type-3** components is $O(\sqrt{n})$.

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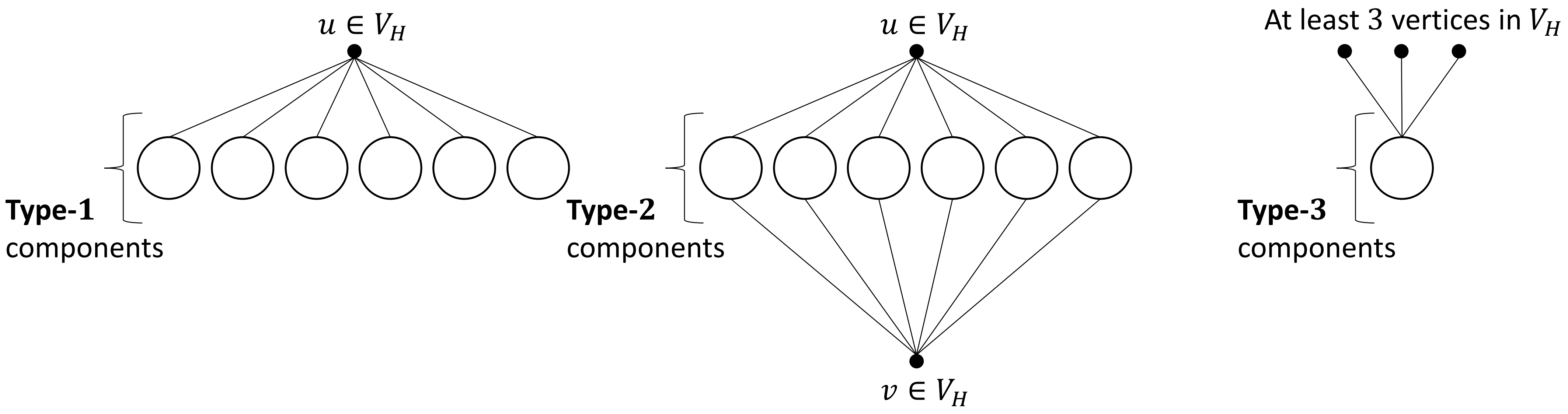
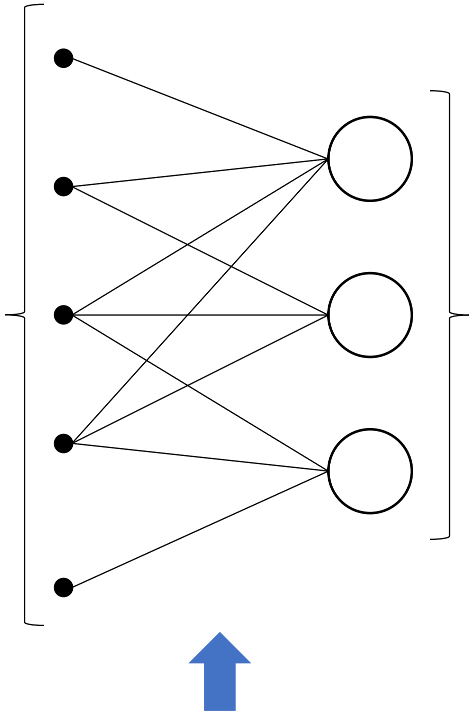


The number of type-3 components

Claim: The number of **type-3** components is $O(\sqrt{n})$.

$Y = V_H$



$X =$ set of type-3 components

$G^* = (V^* = X \cup Y, E^*)$

The genus of G^* is $g = O(1)$.

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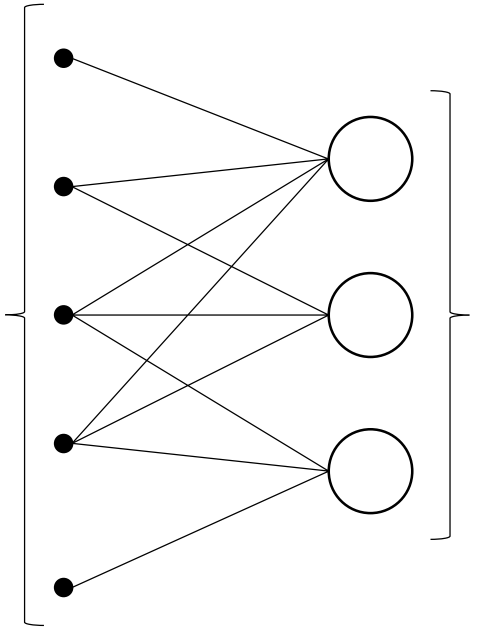

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The genus of G^* is $g = O(1)$.

Lemma: $2|V^*| - |E^*| \geq 4(1-g)$

- Consider a crossing-free drawing of G^* into a surface of genus g.
- Let F^* be the set of faces.
- $2|V^*| - |E^*| \geq 2|V^*| - 2|E^*| + 2|F^*| \geq 4(1-g)$.

Each edge appears in at most two faces:

- $|E^*| \geq 2|F^*|$

Euler's polyhedral formula:

- $|V^*| - |E^*| + |F^*| \geq 2(1-g)$

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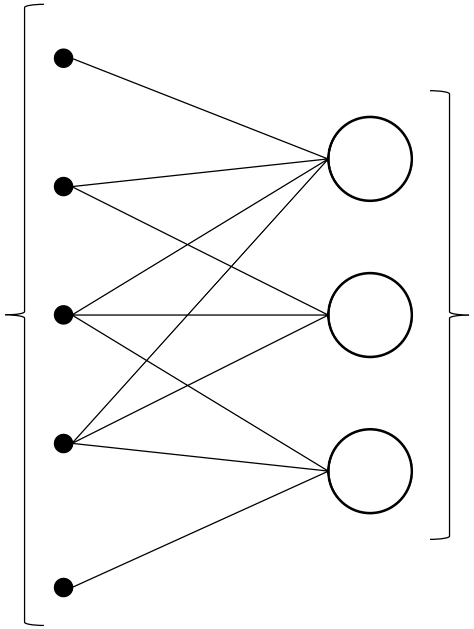

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Lemma: $2|V^*| - |E^*| \leq 2|Y| - |X|$

- $2|V^*| - |E^*| = 2(|X| + |Y|) - |E^*| \leq 2|Y| - |X|$

The degree of each vertex in X is at least 3:

- $|E^*| \geq 3|X|$

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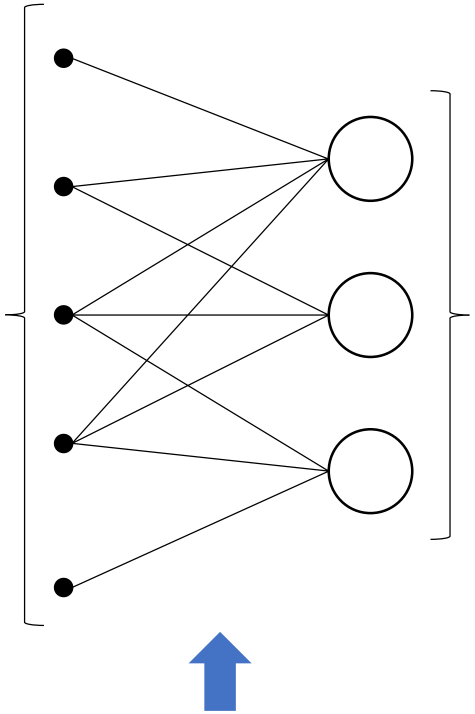

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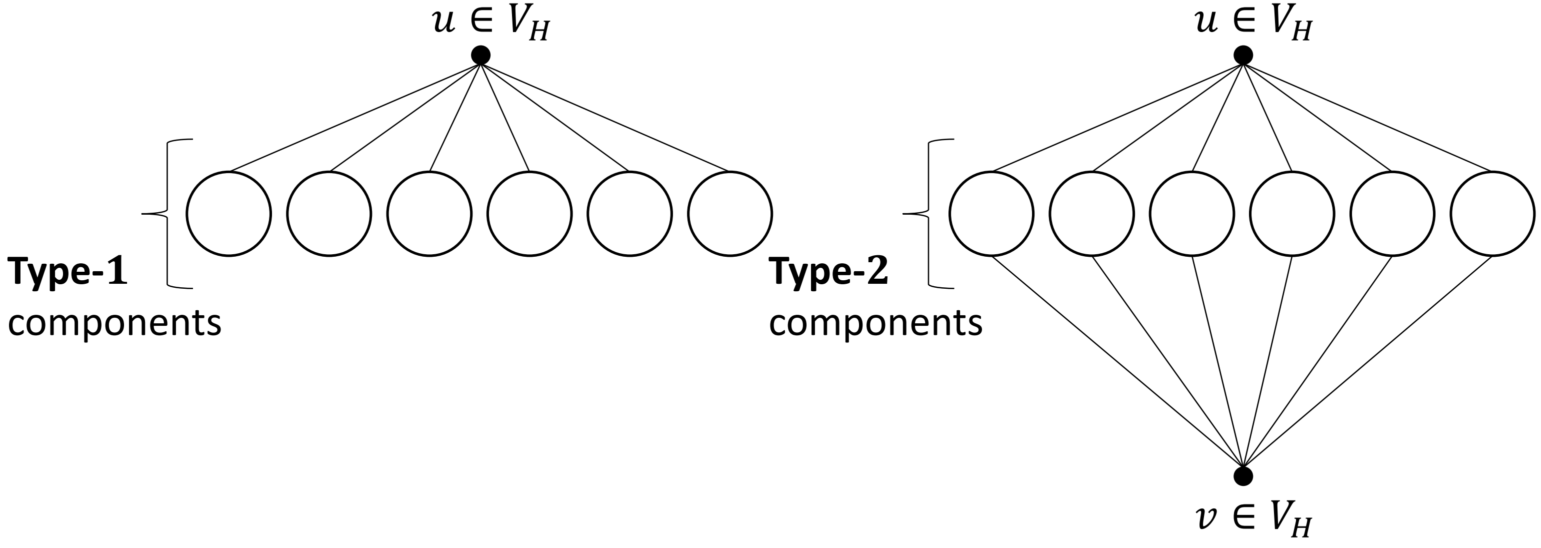
$|X| \leq 2|Y| + 4(1-g) = O(\sqrt{n})$

$G^* = (V^*, E^*)$

The genus of G^* is $g = O(1)$.

The remaining components

- It remains to deal with type-1 and type-2 components.
- **Idea:** To solve the considered problems, it is sufficient to extract a small quantity of information from these components.



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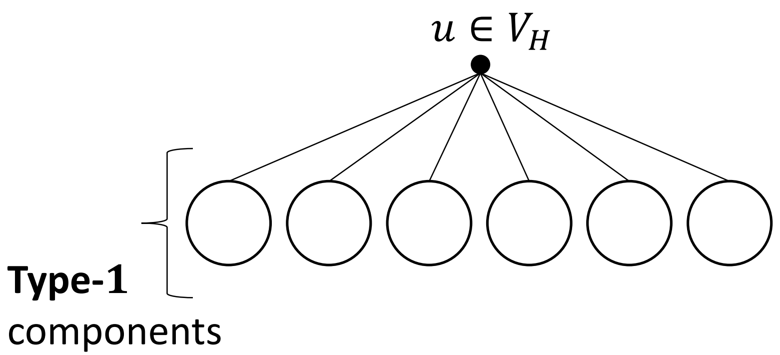

For example, consider the **minimum cut** problem:

- For each type-1 component S adjacent to $u \in V_H$:
  - $c(S) =$ the minimum cut size of $G[S \cup \{u\}]$.
- The only information $u \in V_H$ needs to acquire from its adjacent type-1 components:
  - The minimum value of $c(S)$ ranging over all type-1 components S adjacent to u.
  - This number can be calculated efficiently in the radio network model.

Conclusions

- For bounded-genus graphs, the following problems can be solved with $\tilde{O}(\sqrt{n})$ energy in radio networks:
  - Diameter computation.
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- **Future directions:**
  - Can we achieve the same results with small message sizes?
  - Improve the energy complexity $\tilde{O}(\sqrt{n})$.
  - Other graph classes, e.g., the unit-disc graphs.
  - The energy complexity in other models, e.g., LOCAL and CONGEST.

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