Distributed coloring of hypergraphs

Duncan Adamson [1], Magnús M. Halldórsson [2], and Alexandre Nolin [3]

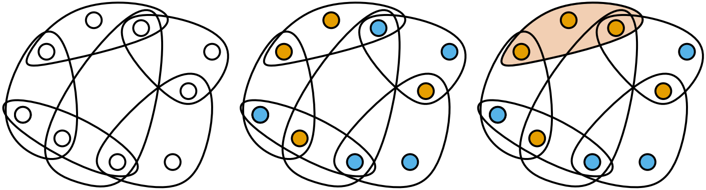
[1] University of Liverpool [2] Reykjavik University [3] CISPA

SIROCCO 2023

Setting

- Focus on LOCAL during the talk.
  - communication network = (hyper)graph to color
  - synchronous message-passing of arbitrary size
  - n nodes, maximum degree Δ

  (But ideas mentioned during the talk are relevant to other models (CONGESTED CLIQUE, streaming...)
- Hypergraph coloring: no edge should be monochromatic.



Randomized distributed graph coloring

Upper bounds:

- poly log log n algorithms for many versions ($\Delta + 1$ coloring [CLP SIAM J.COMP'20], degree+1 list-coloring [HKNT STOC'22], Δ-coloring [FHM SODA'23])
- $O(\log^* n)$ algorithms when each node has access to $\Omega(\text{poly}(\log n))$ colors [SW PODC'10].

Lower bounds:

- $\Omega(\log_\Delta \log n)$ for Δ-coloring.[BFHKLRSU STOC'16]
- $\Omega(\log^* n - \log K)$ for coloring with K colors.

Randomized distributed graph coloring

Upper bounds:

- poly log log n algorithms for many versions ($\Delta + 1$ coloring [CLP SIAM J.COMP'20], degree+1 list-coloring [HKNT STOC'22], Δ-coloring [FHM SODA'23])
- $O(\log^* n)$ algorithms when each node has access to $\Omega(\text{poly}(\log n))$ colors [SW PODC'10].

Lower bounds:

- $\Omega(\log_\Delta \log n)$ for Δ-coloring.[BFHKLRSU STOC'16]
- $\Omega(\log^* n - \log K)$ for coloring with K colors.

Kind of tight for Δ-coloring

Randomized distributed graph coloring

Upper bounds:

- poly log log n algorithms for many versions ($\Delta + 1$ coloring [CLP SIAM J.COMP'20], degree+1 list-coloring [HKNT STOC'22], Δ-coloring [FHM SODA'23])
- $O(\log^* n)$ algorithms when each node has access to $\Omega(\text{poly}(\log n))$ colors [SW PODC'10].

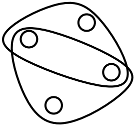
Lower bounds:

- $\Omega(\log_\Delta \log n)$ for Δ-coloring.[BFHKLRSU STOC'16]
- $\Omega(\log^* n - \log K)$ for coloring with K colors.

Kind of tight for Δ-coloring
Much less so when the number of colors is more relaxed

How many colors do hypergraphs need?

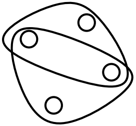
- Suppose all edges of G contain r nodes (r-uniform)
  $\Rightarrow G$ is $O(\Delta^{1/(r-1)})$-colorable.[EL Coll. Math Soc. J. Bolyai'73]
- Suppose G furthermore is linear (edges share ≤ 1 nodes)

  (no such configuration: )

  $\Rightarrow G$ is $O((\Delta/\log\Delta)^{1/(r-1)})$-colorable.[FM J. Comb. Theory B'13]

How many colors do hypergraphs need?

- Suppose all edges of G contain r nodes (r-uniform)
  $\Rightarrow G$ is $O(\Delta^{1/(r-1)})$-colorable.[EL Coll. Math Soc. J. Bolyai'73]
- Suppose G furthermore is linear (edges share ≤ 1 nodes)

  (no such configuration: )

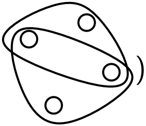
  $\Rightarrow G$ is $O((\Delta/\log\Delta)^{1/(r-1)})$-colorable.[FM J. Comb. Theory B'13]

Lower bound: even Δ-coloring takes $\Omega(\log_{r\Delta}\log n)$ rounds

[BBKO SODA'23]

How many colors do hypergraphs need?

- Suppose all edges of G contain r nodes (r-uniform)
  $\Rightarrow G$ is $O(\Delta^{1/(r-1)})$-colorable.[EL Coll. Math Soc. J. Bolyai'73]
- Suppose G furthermore is linear (edges share ≤ 1 nodes)

  (no such configuration: )

  $\Rightarrow G$ is $O((\Delta/\log\Delta)^{1/(r-1)})$-colorable.[FM J. Comb. Theory B'13]

Lower bound: even Δ-coloring takes $\Omega(\log_{r\Delta}\log n)$ rounds

[BBKO SODA'23]

Lower bound holds on hypertrees, so also for linear hypergraphs

Our results

Theorem

There is a randomized LOCAL *algorithm for* $O(\Delta^{1/(r-1)})$*-coloring* r*-uniform hypergraphs, w.h.p., in* $\operatorname{poly}\log\log n$ *rounds.*

Kind-of-tight: coloring hypergraphs has randomized complexity $\log^{\Theta(1)} \log n$ for any number of colors between Δ and $O(\Delta^{1/(r-1)})$.

Our results

Theorem

There is a randomized LOCAL *algorithm for* $O(\Delta^{1/(r-1)})$*-coloring* r*-uniform hypergraphs, w.h.p., in* $\operatorname{poly}\log\log n$ *rounds.*

Kind-of-tight: coloring hypergraphs has randomized complexity $\log^{\Theta(1)}\log n$ for any number of colors between Δ and $O(\Delta^{1/(r-1)})$.

(Also: CONGESTED CLIQUE, Streaming)

Lovász Local Lemma (LLL)

How do we even know that so few colors are needed?

Lovász Local Lemma (LLL)

How do we even know that so few colors are needed? **LLL**

Lovász Local Lemma (LLL)

How do we even know that so few colors are needed? **LLL**

Theorem (Lovász Local Lemma)

Let $\mathcal{E}$ be set of a events s.t.:

- $\forall A \in \mathcal{E}, \Pr[A] \leq p$
- *Each $A \in \mathcal{E}$ is independent from a subset of $\mathcal{E}$ of size $\geq |\mathcal{E}| - d$.*

If $4pd \leq 1$, with probability > 0, none of the events occur.

Lovász Local Lemma (LLL)

How do we even know that so few colors are needed? **LLL**

Theorem (Lovász Local Lemma)

Let $\mathcal{E}$ be set of a events s.t.:

- $\forall A \in \mathcal{E}, \Pr[A] \leq p$
- *Each $A \in \mathcal{E}$ is independent from a subset of $\mathcal{E}$ of size $\geq |\mathcal{E}| - d$.*

If $4pd \leq 1$, with probability > 0, none of the events occur.

Intuitively: even with bad events which are too many/likely to argue by union bound that they don't all occur at one w.p. > 0, it can still be argued if the events are mostly independent.

LLL for coloring hypergraphs

Suppose each node tries a random color u.a.r in $[K]$.

Probability than an edge is monochromatic?

Define the event "e monochromatic" for each edge. How many other events is it dependent with?

How large do we need K so $4pd \leq 1$ to apply LLL?

LLL for coloring hypergraphs

Suppose each node tries a random color u.a.r in $[K]$.

Probability than an edge is monochromatic?

$\Rightarrow \frac{1}{K^{r-1}}$

Define the event "e monochromatic" for each edge. How many other events is it dependent with?

How large do we need K so $4pd \leq 1$ to apply LLL?

LLL for coloring hypergraphs

Suppose each node tries a random color u.a.r in $[K]$.

Probability than an edge is monochromatic?

$\Rightarrow \frac{1}{K^{r-1}}$

Define the event "e monochromatic" for each edge. How many other events is it dependent with?

$\Rightarrow r(\Delta - 1)$

How large do we need K so $4pd \leq 1$ to apply LLL?

LLL for coloring hypergraphs

Suppose each node tries a random color u.a.r in $[K]$.

Probability than an edge is monochromatic?

$\Rightarrow \frac{1}{K^{r-1}}$

Define the event "e monochromatic" for each edge. How many other events is it dependent with?

$\Rightarrow r(\Delta - 1)$

How large do we need K so $4pd \leq 1$ to apply LLL?

$\Rightarrow K \in \Theta(\Delta^{1/(r-1)})$ suffices

LLL for coloring hypergraphs

Suppose each node tries a random color u.a.r in $[K]$.

Probability than an edge is monochromatic?
$\Rightarrow \frac{1}{K^{r-1}}$

Define the event "e monochromatic" for each edge. How many other events is it dependent with?
$\Rightarrow r(\Delta - 1)$

How large do we need K so $4pd \leq 1$ to apply LLL?
$\Rightarrow K \in \Theta(\Delta^{1/(r-1)})$ suffices

Question is: how do find the assignment?

Algorithmic LLL on small instances

Doable in $\operatorname{poly}\log\log n$ time: solving a $\operatorname{poly}\log n$-sized instance, deterministically.

Algorithmic LLL on small instances

Doable in $\text{poly} \log \log n$ time: solving a $\text{poly} \log n$-sized instance, deterministically.

General structure of our algorithm (shattering technique):

- color most nodes with a process that succeeds with some probability $1 - \frac{1}{\text{poly}(\Delta)}$ at each node.
- deal with the remaining $\text{poly} \log n$-sized small patches of uncolored nodes with a deterministic algorithm.

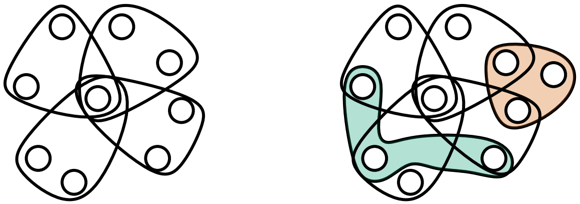
Decreasing the degree

Hope: If each node tries one of K colors, the degree of each node decreases by $\frac{1}{K^{r-1}}$ in expectation, hopefully w.h.p. while the degree is $\Omega(\log n)$.

Decreasing the degree

Hope: If each node tries one of K colors, the degree of each node decreases by $\frac{1}{K^{r-1}}$ in expectation, hopefully w.h.p. while the degree is $\Omega(\log n)$.

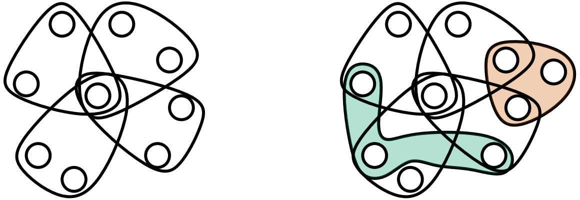
Problem: edges around a node are not independent.



Decreasing the degree

Hope: If each node tries one of K colors, the degree of each node decreases by $\frac{1}{K^{r-1}}$ in expectation, hopefully w.h.p. while the degree is $\Omega(\log n)$.

Problem: edges around a node are not independent.



Solution: Focus on triangle-free hypergraphs first, then reduce to them.

Triangle-free case

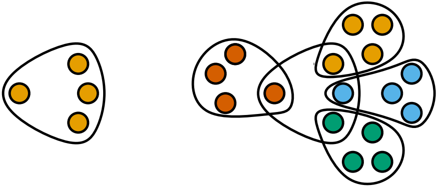
Suppose we try from a new fresh set of colors in each round.

How does an edge survive?

Triangle-free case

Suppose we try from a new fresh set of colors in each round.

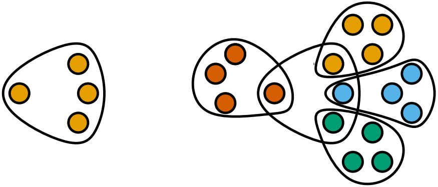
How does an edge survive?



Triangle-free case

Suppose we try from a new fresh set of colors in each round.

How does an edge survive?



Analysis, argue that:

1. few incident edges have their nodes all take the same color as the central node, w.h.p. (independent)
2. not too many colors get "blocked" by distance-2 neighbors.
3. few edges incident to the central node survive by each member choosing a blocked color. (independent once conditioned on item 2)

Reducing to small, low-degree instances

- Set $K = \Theta(\Delta^{1/(r-1)})$
- Repeat for $\Theta(\log \log n)$ rounds:
  - Each node tries a random color in a fresh set of K colors
  - Monochromatic edges uncolor their nodes
  - Each edges with at least one colored node are removed
  - Decrease K by a multiplicative constant factor $\alpha \in [0, 1]$

Reducing to small, low-degree instances

- Set $K = \Theta(\Delta^{1/(r-1)})$
- Repeat for $\Theta(\log \log n)$ rounds:
  - Each node tries a random color in a fresh set of K colors
  - Monochromatic edges uncolor their nodes
  - Each edges with at least one colored node are removed
  - Decrease K by a multiplicative constant factor $\alpha \in [0, 1]$

Total number of colors used is $\Theta(\Delta^{1/(r-1)})$ due to geometric decrease. $O(\log \log n)$ rounds suffice because the number of colors decreases slower than the degree, so success probability increases.

Reducing to small, low-degree instances

- Set $K = \Theta(\Delta^{1/(r-1)})$
- Repeat for $\Theta(\log \log n)$ rounds:
  - Each node tries a random color in a fresh set of K colors
  - Monochromatic edges uncolor their nodes
  - Each edges with at least one colored node are removed
  - Decrease K by a multiplicative constant factor $\alpha \in [0, 1]$

Total number of colors used is $\Theta(\Delta^{1/(r-1)})$ due to geometric decrease. $O(\log \log n)$ rounds suffice because the number of colors decreases slower than the degree, so success probability increases.

This reduces every degree to $O(\log n)$, uncolored nodes form $\operatorname{poly} \log n$ sized components. Finish with deterministic algorithm.

Reducing to triangle-free case

Intuitive idea: put nodes into buckets. Distinct buckets use distinct colors. Do the bucketing such that triangles disappear.

Reducing to triangle-free case

Intuitive idea: put nodes into buckets. Distinct buckets use distinct colors. Do the bucketing such that triangles disappear.

$\Rightarrow$ Reduces the problem to solving multiple triangle-free instances in parallel.

Reducing to triangle-free case

Intuitive idea: put nodes into buckets. Distinct buckets use distinct colors. Do the bucketing such that triangles disappear.

$\Rightarrow$ Reduces the problem to solving multiple triangle-free instances in parallel.

Bucketing works w.h.p. at first, needs LLL once $o(\log n)$ edges ares expected to survive in each bucket.

Reducing to triangle-free case

Intuitive idea: put nodes into buckets. Distinct buckets use distinct colors. Do the bucketing such that triangles disappear.

$\Rightarrow$ Reduces the problem to solving multiple triangle-free instances in parallel.

Bucketing works w.h.p. at first, needs LLL once $o(\log n)$ edges ares expected to survive in each bucket.

(Same strategy used in the proof for the lower chromatic number of linear hypergraphs[FM J. Comb. Theory B'13], also similar to the proof that triangle-free graphs have chromatic number $O(\Delta/\log\Delta)$)

Open questions

- What is the complexity of $O((\Delta/\log\Delta)^{r-1})$-coloring linear hypergraphs? Can you show a stronger lower bound?
- Can we extend the techniques to general hypergraphs?
- CONGEST?

Open questions

- What is the complexity of $O((\Delta/\log\Delta)^{r-1})$-coloring linear hypergraphs? Can you show a stronger lower bound?
- Can we extend the techniques to general hypergraphs?
- CONGEST?

Thanks!